A Survey of Seiberg-Witten Theory and its Applications to 4-manifolds

by

CHAN, Kai Leung

A Thesis Submitted in Partial Fulfilment
of the Requirements for the Degree of
Master of Philosophy
in
Mathematics

August 2007

Thesis/Assessment Committee

Professor LEUNG Nai Chung Conan(Chair)
Professor AU Kwok Keung Thomas (Thesis Supervisor)
Professor WAN Yau Heng Tom (Committee Memeber)
Professor LUO Feng (External Examiner)

Abstract

Seiberg-Witten invariants are diffeomorphic invariants of four dimensional smooth manifolds. After introducing these invariants, some striking results of four dimensional smooth manifolds were given. There are two smooth manifolds with the same topological structure but having different Seiberg-Witten invariants, which means they have different differentiable structures. We will devise Seiberg-Witten invariants and some topological results of smooth four manifolds, and give examples on that.

摘要

Seiberg-Witten不變量是四維光滑流形的不變量。通過引入Seiberg-Witten不變量，我們能更了解四維光滑流形。其中一種應用是計算兩個同胚的四維光滑流形的Seiberg-Witten不變量，得出各異的結果，從而証明這個流形擁有多於一個可微結構。本論文將介紹Seiberg-Witten不變量和關於四維流形的拓撲性質，並且構作一些擁有多於一個可微結構的四維流形的實例。

ACKNOWLEDGMENTS

I would like to express my deepest gratitude to my supervisor Prof. AU Kwok Keung Thomas for his guidance and encouragement in these two years. He is always willing to spend his time on me when I got lost. I would also like to thank my family, colleagues and my best friends Debby K.S. Lam, Byron K.C Ng and C.P. Tam, I was survived by their support when I faced difficulty. Finally, I must thank the mathematics department for providing a nice studying environment to me and giving me a fruitful life.

Contents

Chapter 0

Introduction

A smooth manifold is said to be admitting an exotic structure if it has more than one distinct smooth structures. One of the most famous example was constructed by John Milnor [Mi], he showed that 7-sphere has exactly 28 different smooth structures, and 15 if orientation is omitted. Mathematicans were shocked that even a simple manifold as sphere can carry more than one smooth structure.

Later on, with the discovery of Donaldson Invariants in 4-dimensional manifolds, an exotic $\mathbb{R}^4$ was first found by Robion Kirby and Michael Freedman. Also, it was proved that for any positive integer n other than 4, there are no exotic $\mathbb{R}^n$. In other words, a smooth n-manifold with $n \neq 4$ which is homeomorphic to $\mathbb{R}^n$ is in fact diffeomorphic to $\mathbb{R}^n$.

On the other hand, we knew that a closed n-manifold M^n has exactly one smooth structure for $n \leq 3$ and has at most finitely many smooth structures for $n \geq 5$, also they can be completely classified. However there was not much breakthrough for 4-manifolds until Seiberg-Witten invariants, diffeomorphic invariants for smooth 4-manifolds, were invented. By calculating Seiberg-Witten invariants, researchers got more examples of 4-manifolds with exotic structures. Also surprisingly, mathematican obtained the following results:

1. there exist topological 4-manifold which cannot be "smoothened".

2. no smooth 4-manifolds with only finitely many distinct smooth structures was found.

3. there exist smooth 4-manifolds with infinitely many distinct smooth structures.

Out of our expectations, the class of smooth 4-manifolds is much complicated. Even finding concrete examples of 4-manifolds with exotic structures is a difficult task as we can see in this survey. It seems that we are still far away from the goal of finding a complete classification.

In this survery, we will mainly focus on 4-manifolds. The first part is about the background. In chapter 1, we will revise some materials, show the ideas of construction of Seiberg-Witten invariants and some important results about Seiberg-Witten invariants. In chapter 2 and 3, we will revise some tools to study 4-manifolds such as intersection forms and Kirby Calculus. By using those tools, we can study some new techniques to construct some special 4-manifolds which are useful in later parts.

The second part is about examples of 4-manifolds having exotic structures. In chapter 4, we will focus on the type $m\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$. By studying deeply to the example $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$, we will state the ideas and difficulties to construct exotic sturctures for this kind of 4-manifolds, and finally revise the progress of researches in this area. In chapter 5, we will visit the Seiberg-Witten invariants again. This time we will consider all Seiberg-Witten invariants at once to formulate the Seiberg-Witten Series which is again a diffeomorphic invariant. The advantage of considering Seiberg-Witten Series is that we are able to obtain nice formulas of the series for some operations and surgeries. Finally, knot surgery technique invented by R. Fintushel and R. Stern will be introduced and we will see how it helps us to find examples of 4-manifolds with infinitely many exotic structures.

Part I

Background Scenery

Chapter 1

Seiberg-Witten Invariants

1.1 Preliminaries

In order to construct the Seiberg-Witten invariants, we have to review some materials which can be found in [J], [M], [N].

First of all, we need to know the group $Spin(4)$ group is isomorphic to the product of two pieces of $SU(2), i.e.$

$$Spin(4) = SU(2) \times SU(2)$$

and the group $Spin^c(4)$ can be identified as following:

$$Spin^c(4) = \left\{ \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} : A_+, A_- \in SU(2), \lambda \in U(1) \right\}$$

Actually, spin group and complex spin group can be defined by introducing Clifford Algebra and complex Clifford Algebra (refer section 1.8 of [JJ]), we will have a short introduction later in this section.

If we identify $\mathbb{R}^4$ with subspace of 2×2 complex matrices via the map

$$(t, x, y, z) \mapsto \begin{pmatrix} t + iz & -x + iy \\ x + iy & t - iz \end{pmatrix}$$

then we have

$$\begin{aligned} \|(t,x,y,z)\|^2 &= t^2+x^2+y^2+z^2 \\ &= det(Q) \end{aligned}$$

We define a group homomorphism $\rho : Spin^c(4) \longrightarrow GL(\mathbb{R}^4)$ by the formula

$$\rho \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} (Q) = (\lambda A_-)Q(\lambda A_+)^{-1}$$

where $Q \in \mathbb{R}^4$ and represented as a 2×2 complex matrix.

$$\begin{aligned} det[(\lambda A_-)Q(\lambda A_+)^{-1}] &= det(A_-) \cdot det(Q) \cdot det(A_+) \\ &= det(Q) \end{aligned}$$

We can see that $\rho \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix}$ is an element in $GL(V)$ which preserves length and so the image of the group homomorphism ρ is in $SO(4)$.

Also we have another group homomorphism $\pi : Spin^c(4) \longrightarrow U(1)$ defined by

$$\pi \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} = \lambda^2$$

Now suppose M is a Riemannian 4-manifold, the Riemannian metric enable us to reduce the structure group of the tangent bundle TM to $SO(4)$. Therefore, we can choose a trivializing open cover $\{U_\alpha : \alpha \in A\}$ for M so that the corresponding transition functions take their values in $SO(4)$:

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow SO(4)$$

Definition 1.1.1. A *spin*^{c} structure on M is given by an open covering $\{U_\alpha : \alpha \in A\}$ and a collection of transition functions

$$\widetilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow Spin^c(4)$$

such that $\rho \circ \widetilde{g}_{\alpha\beta} = g_{\alpha\beta}$ and cocycle conditions are satisfied.

From a $spin^c$ structure, we can construct a principal $U(1)$ bundle by the composed function $\pi \circ \widetilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow U(1)$. Then we can associate a complex line bundle with $U(1)$ as structure group. The reason why we use the projection map π such that the image is given by λ^2 instead of λ is that the former one is a well defined homomorphism and so it can give a well defined line bundle, but not for the latter one. As a result, we denote the complex line bundle constructed above as L^2 and call L a virtual complex line bundle.

Similarly, by considering the left upper block and the right lower block matrix, we get two rank 2 complex bundles which are called positive and negative spinor bundle, and denoted by $S_+ \otimes L$ and $S_- \otimes L$. When we consider the whole matrix, the complex bundle is called the (twisted) spinor bundle, denoted by $S \otimes L$, and so we have $S \otimes L = S_+ \otimes L \oplus S_- \otimes L$. The preimage of any point on the manifold M of the map $\pi : S \otimes L \longrightarrow M$ is a vector space which is called spinor space. Similarly, we have positive and negative spinor space.

Now, we want to see how the Levi-Civita connection on the tangent bundle induces a connection on the spinor bundle. Recall that the structure group of the tangent bundle of a n-manifold is $SO(n)$ and a connection on the tangent bundle can be locally expressed as

$$d + \Omega$$

where Ω is 1-form valued in $so(n)$. We know that the structure group of the spinor bundle constructed above is $Spin^c(4)$, in order to construct a connection on it, we need to know the Lie Algebra $spin^c(4)$ of $Spin^c(4)$ and how elements of $spin^c(4)$ act on sections of the spinor space. As we said before, to study the groups $Spin(4)$, $Spin^c(4)$ and the Lie Algebras $spin(4)$, $spin^c(4)$ carefully, we have to introduce Clifford Algebra.

Suppose $e_1, \cdots, e_n$ be an orthonormal basis for a n-dimensional real vector space

V, then Clifford algebra of V is the real vector space denoted by $Cl(V)$ (or $Cl(n)$) which has vectors $1, e_1, \cdots, e_n, e_1 \cdot e_2, e_1 \cdot e_3, \cdots, e_{n-1} \cdot e_n, \cdots, e_1 \cdot e_2 \cdots e_n$ as an orthonormal basis (therefore its dimension $= 2^n$). Also its multiplication is called Clifford multiplication which can be described by the following rules:

$$\begin{aligned} e_i \cdot e_j &= -e_j \cdot e_i \qquad \text{for } i \neq j \\ e_i \cdot e_i &= -1 \end{aligned}$$

Also the complexified Clifford algebra can be defined as

$$Cl^c(V) = Cl(V) \otimes_{\mathbb{R}} \mathbb{C}$$

Furthermore, we can prove some facts:

- $\{a = a_1 \cdots a_{2m} | a \in V, |a_i|^2 = 1 \text{ for all } 1 \leq i \leq 2m\}$ is a group which is defined as spin group $Spin(V)$.

- The subspace spanned by $e_i \cdot e_j$ for $i < j$ forms a Lie algebra with the bracket

$$[a, b] = a \cdot b - b \cdot a$$

  which is defined to be the Lie algebra $spin(V)$.

- $Spin(V)$ is actually a Lie group with Lie algebra $spin(V)$.

- $Cl^c(4)$ is isomorphic to the endomorphism group of the spinor space. (refer to section 1.8 of [JJ] for the construction of this isomorphism.)

Finally, the group $Spin^c(V)$ is defined as the group generated by $Spin(V)$ and elements in $\mathbb{C}$ with length 1.

Note that $spin(4) \subseteq Cl^c(4)$ and the last item above. Under the isomorphism between $Cl^c(4)$ and the endomorphism group of the spinor bundle, we will know how an element of $spin(4)$ acts on a section of spinor bundle. In the language of Clifford Algebra, the action is just Clifford multiplication. Under the isomorphism above, elements e_i for $1 \leq i \leq 4$ are identified as endomorphisms of spinor bundles as followings:

$$e_1 \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} , \quad e_2 \mapsto \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

$$e_3 \mapsto \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} , \quad e_4 \mapsto \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

If we take the above identification and calculate the images of the elements in $spin(4)$, we will find they are matrices in form of

$$\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}$$

and hence an endomorphism which comes from the image of an element of $spin(4)$ preserves the positive and negative spinor space. Furthermore, taking the same identification again, $Spin(4)$ can be regarded as subspace of $\mathbb{C}^{4\times 4}$ and we will discover that the image of $Spin(4)$ is exactly

$$\left\{ \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} : A_+, A_- \in SU(2) \right\}$$

Therefore, the description of $Spin(4)$ at the beginning is just matrix representation and similar for $Spin^c(4)$.

As a matter of course, using matrix representations for those groups can help us to continue studying without the knowledge of Clifford algebra or carrying out complicated calculations in form of matrix multiplications, but defining those groups by introducing Clifford algebra can simplifies the notations and let us to have a more general pictures of spin groups.

Then we get back to the way of construction of a connection on spinor bundle. Firstly, it is a fact that the spin group $Spin(4)$ is double cover of the group $SO(4)$ (actually, not just in 4 dimension), therefore the Lie algebra of them are isomorphic, i.e. $spin(4) \cong so(4)$. Furthermore, the Lie algebra of $Spin^c(4)$ is $spin^c(4) = spin(4) \oplus u(1)$ where $u(1)$ is the Lie algebra the of Lie group $U(1)$.

For the $spin(4)$ part, it is induced from the Levi-Civita connection. Locally the Levi-Civita connection can be decomposed as

$$\begin{aligned} \nabla &= d + \Omega \quad \text{where } \Omega \text{ is one form valued in } so(4) \\ &= d + \sum_{i<j} \Omega_{ij} e_i \wedge e_j \end{aligned}$$

where Ω_{ij} are one forms and $e_i \wedge e_j$ denotes the matrix with -1 at the (i,j)-entry, 1 at the (j,i)-entry and 0 otherwise. The isomorphism between $so(4)$ and $spin(4)$ is given by

$$e_i \wedge e_j \mapsto \frac{1}{2} e_i \cdot e_j$$

where the product on the right hand side is the Clifford multiplication. By using this isomorphism and specifying the $u(1)$ part, i.e. choosing a unitary connection A, it is possible to define a connection on the spinor bundle as following:

$$\nabla^A = d + \frac{1}{2}(\sum_{i<j} \Omega_{ij} e_i \cdot e_j + ia)$$

(Since $u(1)$ can be identified with $i\mathbb{R}$, we may write A as $d + ia$ where a is a real valued 1-form.)

Finally, we come to the definition of the Dirac operator corresponding to unitary connection A.

Definition 1.1.2. The Dirac operator $D_A : \Gamma(S \otimes L) \longrightarrow \Gamma(S \otimes L)$ corresponding to unitary connection A is defined by

$$D_A(\psi) = \sum_{i=1}^{4} e_i \cdot \nabla_{e_i}^A \psi$$

One of the most important result is that the Dirac operator D_A maps sections in the positive spinor bundle (sometimes sections of positive spinor bundle are called positive spinor fields, and so for negative one) to negative one and maps sections in the negative spinor bundle to positive one. Therefore we denote them as

$$D_A^+ : \Gamma(S_+ \otimes L) \longrightarrow \Gamma(S_- \otimes L)$$
$$D_A^- : \Gamma(S_- \otimes L) \longrightarrow \Gamma(S_+ \otimes L)$$

There are so many properties about the Clifford Algebra, *spin*^{c} structures and Dirac operators. Some cannot be covered here, but they can be found in the references mentioned before. We will state them when we encountered in the coming context.

1.2 Construction of Seiberg-Witten Invariants

Now we come to the construction of Seiberg-Witten invariants. Suppose M is a four-dimensional Riemannian manifold with a *spin*^{c} structure. From the given *spin*^{c} structure, one can associate a virtual complex line bundle L and the spinor bundle $S \otimes L$.

1.2.1 Seiberg-Witten Equations and the Moduli Space

1. Seiberg-Witten equations (perturbed with respect to ϕ):

   Firstly, we write down the Seiberg-Witten equations:

$$\begin{aligned} D_A^+\psi &= 0 \\ F_A^+ &= \sigma(\psi) + \phi \end{aligned}$$

   where D_A^+ is the Dirac operator associated by a unitary connection which maps positive spinor fields to negative spinor fields, F_A^+ is the self-dual part of the curvature (we write $-iF_A = \Omega_A = \nabla_A \circ \nabla_A$), ϕ is a given self-dual two form and σ is a quadratic term in ψ which can be written explicitly:

$$\sigma(\psi)(e_i, e_j) = -\frac{i}{2}\langle \psi, e_i \cdot e_j \cdot \psi \rangle$$

where $\langle \cdot, \cdot \rangle$ is the hermitian product of the spinor bundle. We want to find a pair (A, ψ) in the configuration space

$$\mathcal{A} = \{(A, \psi) | A \text{ is a unitary connection on } L^2,\ \psi \in \Gamma(S^+ \otimes L)\}$$

such that the two Seiberg-Witten equations are satisfied.

2. Solution space:

   Let $\mathcal{G}$ be the set of all smooth map from M to S^1 which is the gauge group. Suppose that $g \in \mathcal{G}$, we can check that if (A, ψ) satisfies the Seiberg-Witten equations, then g acts on (A, ψ) also satisfies the Seiberg-Witten equations, i.e.

   $$g \cdot (A, \psi) = (gAg^{-1}, g\psi)$$

   satisfies the Seiberg-Witten equations. Therefore

   $$\mathcal{M}_\phi = \{[A, \psi] \in \mathcal{B} | (A, \psi) \text{ satisfies the SW equations}\}$$

   is well-defined where $\mathcal{B} = \mathcal{A}/\mathcal{G}$.

After that, we claim that for generic choice of ϕ and Riemannian metric g, the solution set $\mathcal{M}_\phi$ is a finite dimensional, compact and oriented manifold. Furthermore, the formal dimension of the solution space is

$$2(\text{index of } D_A^+) - b_2^+ - 1 = \frac{1}{4}(c_1^2(L) - 2\chi(M) - 3\sigma(M))$$

where $\chi(M)$ is the Euler characteristic of M and $\sigma(M)$ is the signature of intersection form of M which will be explained in next chapter.

Next, we set

$$\begin{aligned}
\mathcal{A}^* &= \{(A,\psi)\in\mathcal{A}|\psi\neq 0\}, \\
\mathcal{G}_0 &= \{g\in\mathcal{G}|g(p_0)=1\} \quad \text{where } p_0 \text{ is a fixed point on } M, \\
\widetilde{\mathcal{B}^*} &= \mathcal{A}^*/\mathcal{G}_0, \\
\mathcal{B}^* &= \mathcal{A}^*/\mathcal{G}.
\end{aligned}$$

Then, $\widetilde{\mathcal{B}^*}\to\mathcal{B}^*$ can be regarded as a S^1-bundle.

1.2.2 Seiberg-Witten Invariants

From the S^1-bundle constructed above, we can associate a corresponding complex line bundle and the first Chern class of this complex line bundle which is in $H^2(\mathcal{B}^*)$, denoted by c_1.

Assume there exists a generic path ϕ_t joining ϕ_1 and ϕ_2, we can show that there exists an oriented smooth submanifold $\mathcal{W}\subseteq H^2_+(M)\times\mathcal{B}^*$ (where $H^2_+(M)$ is the space of self-dual 2-forms of the manifold M) such that

$$\partial\mathcal{W}=\mathcal{M}_{\phi_1}-\mathcal{M}_{\phi_2}$$

Furthermore, assume the dimension of $\mathcal{M}_{\phi_t}$ is even and let

$$d=\frac{1}{2}\mathrm{dim}(\mathcal{M}_{\phi_t})$$

Then c_1^d is a closed $2d$-differential form, i.e. $c_1^d=d\beta$ for some $(2d-1)$-form β.

Therefore we have the following result,

$$\begin{aligned} \langle [c_1^d], \mathcal{M}_{\phi_1} \rangle - \langle [c_1^d], \mathcal{M}_{\phi_2} \rangle &= \int_{\mathcal{M}_{\phi_1}} c_1^d - \int_{\mathcal{M}_{\phi_2}} c_1^d \\ &= \int_{\partial \mathcal{W}} c_1^d \\ &= \int_{\partial \mathcal{W}} d\beta \\ &= 0 \\ \therefore \quad \langle [c_1^d], \mathcal{M}_{\phi_1} \rangle &= \langle [c_1^d], \mathcal{M}_{\phi_2} \rangle \end{aligned}$$

Similarly, it was shown that changing the Riemannian metric on M alters the moduli space by a cobordism. Therefore the number $\langle [c_1^d], \mathcal{M}_\phi \rangle$ is an invariant for generic choice of self-dual 2-form ϕ and metric g, hence a diffeomorphic invariant. We denote it as $SW_M(L)$, or most of the time we would like to refer to the *spin*^{c} structure given and denote the invariant by $SW_M(\mathfrak{s})$ where $\mathfrak{s}$ is the *spin*^{c} structure. We will simply write $SW(L)$ if no confusion occurs.

To explain it explicitly, suppose that $f : M_1 \to M_2$ is a diffeomorphism between two smooth manifolds. Given a *spin*^{c} structure $\mathfrak{s}_2$ on M_2, f associates a *spin*^{c} structure $\mathfrak{s}_1 = f^*\mathfrak{s}_2$ on M_1. Then $SW_{M_1}(\mathfrak{s}_1) = SW_{M_2}(\mathfrak{s}_2)$ and hence Seiberg-Witten invariants are diffeomorphic invariants. Furthermore, if $\mathfrak{s}$ is a *spin*^{c} structure such that $SW(\mathfrak{s}) \neq 0$, we call $\mathfrak{s}$ is a basic class.

1.2.3 Remarks

1. *spin*^{c} structure:

   Each oriented four dimensional Riemannian manifold possesses a *spin*^{c} structure.

2. Configuration space:

   If we consider the Sobolev completion (respect to a suitable Sobolev norm) of $\mathcal{A}$, $\widetilde{\mathcal{B}}$ and etc., then Sobolev Embedding Theorem and other nice results from PDE theory can be applied to prove the compactness of the solution space. Of course, we have to make sure that changing the choice of Riemannian metric and related information is just replacing the norm by a equivalent one, and therefore it does not affect the resulting completion.

3. Conditions on b_2^+:

   First of all, b_2^+ should be odd to guarantee $dim(\mathcal{M}_\phi)$ is even, recall that $dim(\mathcal{M}_\phi) = 2(\text{index of } D_A^+) - b_2^+ - 1$.

   Also for the Seiberg-Witten equations, we claim that the reducible solution in $\mathcal{M}_\phi$ can only occur when $\psi = 0$. Recall that

   $$\mathcal{M}_\phi = \{[A,\psi] \in \mathcal{B} | (A,\psi) \text{ satisfies the SW equations}\}$$

   where $\mathcal{A}$ is the configuration space and $\mathcal{B} = \mathcal{A}/\mathcal{G}$. Note that the group $\mathcal{G}$ acts freely on $\mathcal{A}$ except those $(A,0) \in \mathcal{A}$. As a result, reducible solutions can only occur in the quotient space $\mathcal{B}$ if the point is in form of $[A,0]$ and hence our claim holds.

   If $\psi = 0$, it remains to solve the equation

   $$F_A^+ = \phi$$

   However, ϕ can be the self-dual part of the curvature of some unitary connection A if and only if ϕ lies in an affine subspace of codimension b_2^+. Assume $b_2^+ \geq 2$, one can obtain a generic curve mentioned in the last section so that it misses that affine subspace and no reducible solution exists for any ϕ on the curve. As a result, we are able to guarantee that $\mathcal{W}$ a smooth manifold and make the whole arugment works.

   Furthermore, more careful treatment can be made to obtain the Seiberg-Witten invariants for the case $b_2^+ = 1$, and we will cover that in the next subsection.

4. Implicit function theorem:

   In order to show that $\mathcal{W}$ is smooth, we need the infinite dimensional version of Implicit Function Theorem.

5. Virtual complex line bundle:

   This survey follows some texts to denote the complex line bundle constructed from the given *spin*^{c} structure as L^2 and denote the virtual complex line bundle (square root of the former line bundle) as L, while some texts may denote those two as L and $L^{-1/2}$, so it may cause confusion for those encounter at the first time.

1.2.4 Seiberg-Witten Invariants for $b_2^+ = 1$

As we have seen in the last subsection, the affine subspace mentioned before has codimension 1, so it divides the space of self-dual two forms into two parts, called chambers. When we perturb ϕ, the curve joining ϕ_1 and ϕ_2 may hit that affine subspace and may cause singularities in the solution space. Therefore, if we only perturb for a "small" ϕ (in the sense of norm), Seiberg-Witten invariants are still be able to be defined.

First, we fix a metric g. Since the dimension of the space of self-dual harmonic 2-forms $dim(\mathcal{H}_+^{2,g}(M)) = b_2^+ = 1$ (with respect to g), there exist a self-dual 2-form with norm one ω_g in the positive component (preassigned).

Then we investigate the second equation of the Seiberg-Witten equations again, note that $[F_A] = 2\pi c_1(L^2) \in H^{2,g}(M) = \mathcal{H}^{2,g}(M)$, so we have $[F_A^+] = 2\pi(c_1(L^2))^+ \in \mathcal{H}_+^{2,g}(M)$, then:

$$\begin{aligned} F_A^+ &= \sigma(\psi) + \phi \\ 2\pi(c_1(L))^+ - [\phi] &= [\sigma(\psi)] \end{aligned}$$

We can check whether our perturbation of ϕ hits the affine subspace by doing dot product. As long as the sign of $((2\pi(c_1(L^2))^+ - [\phi_t]) \cdot [\omega_g]$ remains unchanged, we are able to define the Seiberg-Witten invariants for that particular g. (Remark: $2\pi(c_1(L^2))^+ \cdot [\omega_g] = 2\pi c_1(L^2) \cdot [\omega_g]$ since the dot product is taken in $\mathcal{H}_+^{2,g}(M)$.)

Moreover, if we only perturb for a "small" ϕ, the sign will be dominated by $2\pi c_1(L^2) \cdot [\omega_g]$. Thus we can define two Seiberg-Witten invariants, one for each chamber.

Furthermore, it sometimes happens that the sign of $2\pi c_1(L^2) \cdot [\omega_g]$ is the same for generic metrics, then there exists only one invariant obtained by a small perturbation of ϕ and this invariant obtained is independent from the choice of metric g. We denote the the invariant by $SW_M^+(L)$ if $((2\pi(c_1(L^2))^+ - [\phi_t]) \cdot [\omega_g] > 0$ and $SW_M^-(L)$ if $((2\pi(c_1(L^2))^+ - [\phi_t]) \cdot [\omega_g] < 0$.

1.3 Important Results of Seiberg-Witten Invariants

In this section, some important results about Seiberg-Witten invariants are introduced that turn out to be powerful tools.

1.3.1 Manifolds Admit Positive Scalar Metrics

Suppose that M is a compact four dimensional Riemannian manifold. We say that a manifold M admits a positive scalar metric if it equips with a metric such that the scalar curvature at every point is postive. Then we begin with the following lemma of estimation.

Lemma 1.3.1. *If (A, ψ) is a solution to the Seiberg-Witten equations with $\phi = 0$ and assume $|\psi|$ attains the maximum at point $p \in M$, then*

$$|\psi|^2(p) \leq max\{-\frac{1}{4}R(p), 0\}.$$

Proof. Since p is a maximum for $|\psi|^2$,

$$\begin{aligned} \frac{1}{2}\triangle(|\psi|^2)(p) &\geq 0 \\ -\langle d_A\psi, d_A\psi\rangle(p) + \langle \psi, \triangle^A\psi\rangle(p) &\geq 0 \\ \langle \psi, \triangle^A\psi\rangle(p) &\geq 0 \end{aligned}$$

It follows from the Weitzenböck formula that

$$\begin{aligned} \triangle^A\psi + \frac{R}{4}\psi - \sum_{i<j} F_A(e_i, e_j)(ie_i \cdot e_j \cdot \psi) &= (D_A^+)^2\psi \\ &= 0 \end{aligned}$$

Recall that from the second half of the Seiberg-Witten equations, we have

$$F_A^+(e_i, e_j) = \frac{1}{2}\langle \psi, ie_i \cdot e_j \cdot \psi\rangle$$

and ψ is a positive spinor field, hence

$$F_A(e_i, e_j)\langle \psi, ie_i \cdot e_j \cdot \psi\rangle = F_A^+(e_i, e_j)\langle \psi, ie_i \cdot e_j \cdot \psi\rangle$$

Therefore we obtain

$$\begin{aligned} -\frac{R}{4}|\psi|^2(p) + \sum_{i<j}\langle \psi, iF_A(e_i, e_j)e_i \cdot e_j \cdot \psi\rangle &= \langle \psi, \triangle^A\psi\rangle(p) \\ -\frac{R}{4}|\psi|^2(p) - \sum_{i<j} F_A(e_i, e_j)\langle \psi, ie_i \cdot e_j \cdot \psi\rangle &\geq 0 \\ \text{since } F_A(e_i, e_j) \text{ is purely imaginary} & \\ -\frac{R}{4}|\psi|^2(p) - 2\sum_{i<j}|F_A^+(e_i, e_j)|^2 &\geq 0 \end{aligned}$$

and then

$$-\frac{R(p)}{4}|\psi|^2(p) \geq 2|F_A^+(p)|^2 = |\psi(p)|^4$$

when we divide both sides by $|\psi|^2$, the result follows immediately. $\square$

Theorem 1.3.2. *If an oriented Riemannian manifold with b_2^+ is odd and $b_2^+ \geq 2$ has a Riemannian metric of positive scalar curvature, then all Seiberg-Witten invariants vanish, i.e. $SW(\mathfrak{s}) = 0$ for all spin^c structure $\mathfrak{s}$.*

Proof. From lemma (1.3.1), take $\phi = 0$, then ψ must equal to zero and so the solution space is empty. □

With extra care on perturbations as described in (1.2.4), we can extend this result to the case $b_2^+ = 1$, i.e. if Seiberg-Witten invariants can be defined for small perturbation, they must vanish.

1.3.2 Connected Sums

In last subsection, we have a vanish theorem depending on scalar curvature which is a rather geometrical condition. Now we have a vanish theorem which concern the connected sum of manifolds which is a condition rather topological.

Theorem 1.3.3. *Suppose that the 4-manifold M can be smoothly splits as a connected sum*

$$M = N_1 \# N_2$$

with $b_2^+(N_1)$, $b_2^+(N_2) \geq 1$, then all Seiberg-Witten invariants of M vanish.

Proof. Details of the proof involves the gluing and pasting of monopoles (solution spaces) which are very technical, those materials can be found in chapter 4 of [N], we only give the idea of the proof here.

Connected sum of two 4-manifolds can be thought as cutting out a 4-ball inside each one and result two 4-manifolds with S^3 as boundary. Then we glue a cylinder $S^3 \times [0, 1]$ such that two ends are glued to the two S^3 just obtained. Of course, the gluing operation can be performed smoothly. Also by stretching, the geometry of $N_1 \# N_2$ is dominated by the cylinder and the cylinder can be arraged to be positive scalar curvature. Then, all Seiberg-Witten solutions are vanish on the cylinder and hence every solution for M can be decomposed into solutions in N_1 and N_2. In other words, for each *spin*^{c} structure on M, the solution space for M can be regarded as product of solution spaces of N_1 and N_2,

$$\mathcal{M}_{N_1 \# N_2} = \mathcal{M}_{N_1} \times \mathcal{M}_{N_2}$$

On the other hand, *spin*^{c} structrue on M can also be split in a similar fashion,

$$\mathfrak{s}_{N_1 \# N_2} = \mathfrak{s}_{N_1} \# \mathfrak{s}_{N_2}$$

Then for the corresponding line bundle constructed from the *spin*^{c} structure, we have

$$c_1(L(\mathfrak{s}_{N_1 \# N_2})) = c_1(L(\mathfrak{s}_{N_1})) \# c_1(L(\mathfrak{s}_{N_2})) \text{ and } c_1^2(L(\mathfrak{s}_{N_1 \# N_2})) = c_1^2(L(\mathfrak{s}_{N_1})) + c_1^2(L(\mathfrak{s}_{N_2}))$$

where $L(\mathfrak{s})$ is the line bundle corresponding to *spin*^{c} structure $\mathfrak{s}$ (To be consistent with the whole text, we may use $L^2(\mathfrak{s})$).

For connected sum of manifolds, we have

$$\chi(N_1 \# N_2) = \chi(N_1) + \chi(N_2) - 2 \text{ and } b_2^+(N_1 \# N_2) = b_2^+(N_1) + b_2^+(N_2)$$

Therefore we are able to calculate the formal dimension of $\mathcal{M}_{N_1 \# N_2}$:

$$\begin{aligned}
\dim(\mathcal{M}_{N_1 \# N_2}) &= \frac{1}{4}(c_1^2(L(\mathfrak{s}_{N_1 \# N_2})) - 2\chi(N_1 \# N_2) - 3\sigma(N_1 \# N_2)) \\
&= \frac{1}{4}\{c_1^2(L(\mathfrak{s}_{N_1})) + c_1^2(L(\mathfrak{s}_{N_2})) \\
&\qquad -2(\chi(N_1) + \chi(N_2) - 2) - 3(\sigma(N_1) + \sigma(N_2))\} \\
&= \dim(\mathcal{M}_{N_1}) + \dim(\mathcal{M}_{N_2}) + 1
\end{aligned}$$

Assume $\dim(\mathcal{M}_{N_1 \# N_2}) = 0$, it forces either one of $\dim(\mathcal{M}_{N_1})$ and $\dim(\mathcal{M}_{N_2})$ is negative which means the solution space is empty. Therefore $\mathcal{M}_{N_1 \# N_2}$ is empty and all Seiberg-Witten invariants vanish. For $\dim(\mathcal{M}_{N_1 \# N_2}) > 0$, refer to theorem (4.6.1) of [N]. □

1.3.3 Kähler Surfaces

Before going on discussing the results of Seiberg-Witten invariants, we quickly revise some materials as a preparation, refer to chapter 2 of [M] for details.

Now, if a manifold M admits an almost complex structure with Riemannian metric, then the structure group can be reduced to $U(2)$. We can construct a canonical *spin*^{c} structure on M by the map $j : U(2) \to Spin^c(4)$ defined by

$$j(A) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & det(A) & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{A}} \\ 0 & 0 & & \end{pmatrix}$$

Note that 1, $det(A)$, A are transitions corresponding to the trivial line bundle, anti-canonical line bundle and tangent bundle, so the positive and negative spinor bundles can be identified naturally as followings,

$$\begin{aligned} S_+ \otimes L &\cong \Theta \oplus K^{-1} \\ S_- \otimes L &\cong TM \end{aligned}$$

where Θ denotes the trivial line bundle, K^{-1} is the anti-canonical line bundle and TM is the tangent bundle. On the other hand, the virtual complex line bundle L associated by that *spin*^{c} is the one whose square is the anti-canonical line bundle, i.e. $L^2 = K^{-1}$.

A Kähler manifold automatically admits a natural almost complex structure and hence it has a canoncial *spin*^{c} structure.

Theorem 1.3.4. *Suppose that M is a Kähler surface with $b_2^+ \geq 2$, L is the virtual complex line bundle as described above, then $SW(L) = \pm 1$.*

Proof. We only give the outline of the proof, refer to section 3.7 and 3.8 of [M] for details. In the Kähler case, the Dirac operator corresponding to the $Spin^c(4)$ connection

d_{A_0} can be writen down explicitly as

$$D^+_{A_0} = \sqrt{2}(\overline{\partial} + \overline{\partial}^*)$$

where sections of Θ, K^{-1} and TM are identified with $(0,0)$-forms, $(0,2)$-forms and $(0,1)$-forms respectively.

Furthermore, other $Spin^c(4)$ connection is in form of $d_A = d_{A_0} + ia$ for some real-valued one-form a on M. We write sections of the positive spinor bundle in vector form as

$$\psi = \begin{pmatrix} f \\ g \end{pmatrix}$$

Then we have

$$D^+_A \begin{pmatrix} f \\ 0 \end{pmatrix} = \sqrt{2}(\overline{\partial} f + (ia)_{(0,1)} \cdot f)$$

where $(ia)_{(0,1)}$ denotes the $(0,1)$-component of ia.

By taking $\phi = F^+_{A_0}\omega$ where ω is the Kähler form, the Seiberg-Witten equations become

$$\begin{aligned} D^+_A \psi &= 0 \\ F^+_A &= \sigma(\psi) + F^+_{A_0} + \omega \end{aligned}$$

Then we consider this pair

$$A = A_0, \quad \psi = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}$$

$A = A_0$ implies that $a = 0$, therefore we have

$$\begin{aligned} D^+_A \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} &= \sqrt{2}(\overline{\partial}\sqrt{2}) \\ &= 0 \end{aligned}$$

On the other hand, we have $F_{A_0}^{+} = F_{A}^{+}$. It can be calculated that

$$\sigma\begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} = -\omega$$

and so the second Seiberg-Witten equation is satisfied. As a result, we got a solution for the Seiberg-Witten equations.

By showing the solution just found is unique after taking quotient of the gauge group and that it is nondegenerate, we get that $SW(L) = \pm 1$. $\square$

We have to make a remark here before moving on our discussion. Let $\mathscr{L}_M$ be the set of isomorphism classes of $U(1)$ bundle over a 4-manifold M. Then $\mathscr{L}_M$ is a group with complex multiplication as the group operation, also it is isomorphic to $H^2(M;\mathbb{Z})$. The isomorphism is denote by $c_1 : \mathscr{L}_M \longrightarrow H^2(M;\mathbb{Z})$ which is the first chern class, refer to section 1.4 of [GS]. Therefore, when we consider the Seiberg-Witten invariant of the virtual line bundle L, besides referring to L, we can take the line bundle L^2 can consider the image under the above isomorphism and referring the Seiberg-Witten invariant of a class in $H^2(M;\mathbb{Z})$.

Now, we know the image of the canonical line bundle $\bigwedge^2 T^*M$ under this isomorphism is called the canonical class, which is usually denoted by K_M. Then the image of the anti-canonical line bundle is K_M^{-1}. Therefore, some texts may rewrites the above theorem as $SW(K_M^{-1}) = \pm 1$. On the other hand, with the isomorphism c_1, the image of canonical line bundle and anti-canonical line bundle is called canonical class and anti-canonical class, also they are denoted by the same notations K_M and K_M^{-1} respectively.

Example 1.3.5 (K3 surfaces). A K3 surface is a compact, complex, simply connected surface with trivial canonical line bundle. There are many nice properties about K3 surfaces:

1. All K3 surfaces are diffeomorphic.

2. All K3 surfaces are in fact Kähler manifolds.

3. One description of K3 surface is fiber sum of two $E(1)$, which will be explained in the chapter 5.

With the first property, all K3 surfaces have same Seiberg-Witten invariants and hence we only need to take one representative for calculations. Also the second property tells us that

$$SW_{K3}(K^{-1}) = \pm 1$$

1.3.4 Symplectic Manifolds

The result of Kähler manifolds in the last subsection was first handled by E. Witten [W], by much more afforts, the result was extended to symplectic manifolds by C. Taubes [T1]. A shorter introduction to the theorem appears in note 10.7 of [S].

Theorem 1.3.6. *Let M be a compact, oriented, 4-manifold with $b_2^+ \geq 2$. Let ω be a symplectic form on M with $\omega \wedge \omega$ giving the orientation. Then the virtual complex line bundle of the associated almost complex structure on M has Seiberg-Witten invariant equal to ± 1.*

With careful treatments, this result can be extended to the case $b_2^+ = 1$ and we have $SW_M^-(L) = \pm 1$.

Suppose that M is a symplectic manifold with symplectic structure ω and compatible almost complex structure J. As described before, the almost complex structure J induces a *spin*^{c} structure $\mathfrak{s}_J$. We know that $\{spin^c \text{ structures } \mathfrak{s}\} \cong H^2(M;\mathbb{Z})$, see chapter 10 of [S]. By identifying *spin*^{c} structures and cohomology group and using this $\mathfrak{s}_J$, we can parametrize all other *spin*^{c} structure $\mathfrak{s}$ on M as

$$\mathfrak{s} = \mathfrak{s}_J + \varepsilon \qquad \text{where } \varepsilon \in H^2(M;\mathbb{Z})$$

Let L_J and L_ε be the virtual complex line bundle associated by *spin*^{c} structures $\mathfrak{s}_J$ and $\mathfrak{s}$ respectively. Then we have

$$\begin{aligned} c_1(L_\varepsilon^2) &= c_1(L_J^2) + 2\varepsilon \\ &= K_M^{-1} + 2\varepsilon \qquad (\text{Here, } K_M^{-1} \in H^2(M;\mathbb{Z})) \end{aligned}$$

With the above, Seiberg-Witten invariants can be reformulate as a function:

$$SW_M : \{K_M^{-1} + 2\varepsilon | \varepsilon \in H^2(M;\mathbb{Z})\} \longrightarrow \mathbb{Z}$$

By careful analytical treatments, we have the following lemma:

Lemma 1.3.7. *If $SW_M(K_M^{-1} + 2\varepsilon) \neq 0$, then ε must satisfy the inequalities*

$$0 \leq \varepsilon \cdot [\omega] \leq -K_M^{-1} \cdot [\omega]$$

and the first equality holds only if $\varepsilon = 0$ and the second equality holds only if $\varepsilon = -K_M^{-1}$.

Recall that by definition a K3 surface has trivial canonical line bundle, therefore $K_{K3}^{-1} = 0$. Furthermore, a Kähler manifold is in fact a symplectic manifold. By using the lemma above, the only possibility for $K_M^{-1} + 2\varepsilon$ to be a basic class is $\varepsilon = 0$.

Corollary 1.3.8. *K3 surface has only one basic class which is the anti-canonical class.*

Chapter 2

Intersection Forms

Due to M. Freedman's results in [F] and [FQ], we are able to classify, up to homeomorphism, the smooth simply connected four manifolds by calculating their intersection forms. For our purposes, we only have a short trip on this topic and details can be found in the references mentioned above.

2.1 Intersection Forms of 4-manifolds

Definition 2.1.1. Suppose that M is a compact, oriented, topological four manifold without boundary. The symmetric bilinear form

$$Q_M : H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) \to \mathbb{Z}$$

defined by $Q_M(a,b) = \langle a \cup b, [M] \rangle \in \mathbb{Z}$ is called the intersection form of M and we usually denote it by $a \cdot b$.

By considering the Poincaré duality, we can also define the intersection forms on $H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z})$ by $\langle P.D.(\alpha) \cup P.D.(\beta), [M] \rangle$ where α, $\beta \in H_2(M;\mathbb{Z})$. By considering intersection forms by the above way, we can interpret it as counting the intersection points (with sign) of two surfaces in M. Furthermore, the intersection form defined has a matrix representation, we denote it again by Q_M. For simplicity, we drop

M when there is no confusion.

Before going deeper, we have to recall that for a simply connected 4-manifold, the rank of the zero and fourth cohomology group is the number of connected components and so they are of rank one now; the rank of the first and third cohomology group is counting loops in certain sense, in fact the first cohomology group is isomorphic to the quotient of fundamental group and its commutator subgroup, however the simply connectedness implies the fundamental group is trival and hence the rank of the first and third cohomology group are zero. Therefore the topology of a simply connected manifold is depending on the second cohomology group heavily, and the intersection form is exactly determine how the elements in the second cohomology group interacts, thus it is quite reasonable that it turn out to be an invariant of simply connected 4-manifolds.

A number of terminologies are going to be introduced. Firstly, we define rank $rk(Q)$ to be the dimension of $H^2(M;\mathbb{Z})$, i.e. $rk(Q) = b_2$. By extending $H^2(M;\mathbb{Z})$ to

$$H^2(M;\mathbb{R}) = H^2(M;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$$

and then diagonalize the matrix over $\mathbb{R}$ (or over $\mathbb{Q}$), we can determine the number of positive and negative eigenvalues (counting with multiplicity), so we denote it by b_2^+ and b_2^- respectively. Next, we define the signature $\sigma(Q) = b_2^+ - b_2^-$. Finally, we come to the parity of Q. Q is said to be even if $Q(a,a) \equiv 0 \pmod 2$ for all $a \in H^2(M;\mathbb{Z})$; Q is odd otherwise.

Definition 2.1.2. Q is said to be positive (negative) definite if $rk(Q) = \sigma(Q)$ ($rk(Q) = -\sigma(Q)$). Otherwise, Q is indefinite.

To make it more explicit, Q is positive definite if all eigenvalues are positive.

Definition 2.1.3. A square matrix P is called a unimodular matrix if $det(P) = \pm 1$. A bilinear form is said to be a unimodular form if its matrix representation has determinant ± 1.

For an intersection form Q of a 4-manifold M, it is clear that the matrix representation is symmetric (since $a \cup b = b \cup a$ in 4-dimensional case). Thus, it can be diagonalized and with scaling, we can change a basis so that the matrix representation with respect to this basis has determinant ± 1, i.e. it is unimodular.

2.2 Classification Theorem

Theorem 2.2.1 (M. Freedman, [F], [FQ]). *For each unimodular symmetric bilinear form Q, there exists a simply connected, topological 4-manifold M without boundary such that its intersection form is Q. Furthermore, if Q is even, the manifold is unique up to homeomorphism; if Q is odd, there are exactly two homeomorphic types of manifolds with the given intersection form, but at most one of them carries a smooth structure.*

Consequently, homeomorphic types of simply connected, smooth four manifolds are completely determined by their intersection forms. In particular, for those indefinite unimodular forms, we have the following classifications.

Theorem 2.2.2. *If Q_1 and Q_2 are indefinitite unimodular forms having same rank, signature and parity, then they are equivalent.*

Furthermore, for an indefinite, unimodular form, if the rank, signature and parity are given, then we are able to write down the exact intersection form by the following theorem.

Theorem 2.2.3. *Suppose that Q is an indefinite, unimodular form. If Q is odd, then Q is isomorphic to $b_2^+\langle 1\rangle \oplus b_2^-\langle -1\rangle$; if Q is even, then Q is isomorphic to $\frac{\sigma(Q)}{8}E_8 \oplus$*

$\frac{rk(Q)-|\sigma(Q)|}{2}H$ *where*

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad and \quad E_8 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

2.3 Review: Van Kampen's Theorem

As we have seen that intersection form is an topological invariant for simply connected 4-manifolds, that means if we want to obtain any result from Freedman's Classification Theorem, then we have to show the 4-manifold to be studied should be simply connected, i.e. the fundamental group of it is a trivial group.

In fact, Van Kampen's Theorem is one of the most powerful tools to calculate fundamental group of a manifold. The main idea of it is decomposing a manifold into pieces and telling how the fundamental groups of those pieces and their intersections relate to the fundamental group of the whole manifold. The proof of this theorem appears in chapter 11 of [Mu] and many textbooks about algebraic topology.

Theorem 2.3.1. *Let M be a manifold. Suppose that $M = U \cup V$, where U and V are open in M so that U, V and $U \cap V$ are both path connected, then the group homomorphism*

$$\phi : \pi_1(U) * \pi_1(V) \longrightarrow \pi_1(M)$$

is surjective where $\pi_1(U) * \pi_1(V)$ is the free product of $\pi_1(U)$ and $\pi_1(V)$ and its kernel *is the normal subgroup N generated by all elements of the form $\iota_1(g)\iota_2(g)^{-1}$ where*

$$\iota_1 : \pi_1(U \cap V) \longrightarrow \pi_1(U) \quad \textit{and} \quad \iota_2 : \pi_1(U \cap V) \longrightarrow \pi_1(V)$$

are group homomorphisms induced by natural inclusion maps and $g \in \pi_1(U \cap V)$. As a result, by the First Isomorphism Theorem, $\pi_1(M) \cong (\pi_1(U) * \pi_1(V))/N$.

Furthermore, we decompose M in a nice way so that V is simply connected, then we have the following corollary:

Corollary 2.3.2. *Assume the hypotheses of the preceding theorem together with the assumption that V is simply connected, then*

$$\pi_1(M) \cong \pi_1(U)/N$$

where N is the normal subgroup generated by the image of the homomorphism

$$\iota_1 : \pi_1(U \cap V) \longrightarrow \pi_1(U)$$

The corollary follows immediately simply by noting that $\pi_1(V)$ is trivial and $\iota_2(g) = e$ for all $g \in \pi_1(U \cap V)$ and so $\pi_1(M) \cong \pi_1(U)/N$ and N is the normal subgroup generated by $\iota_1(g)$.

Chapter 3

Kirby Calculus

By breaking down a manifold into pieces according some rules and see how they can glue back the manifold, we can obtain some topological information, such as homology groups, of the manifold. Such breaking down action is called handle decomposition. In particular, for the 4-dimensional case, Ribion Kirby introduced the technique to visualize 4-manifolds by Kirby diagrams, namely drawing the handles of 4-manifolds out and performing handle slides and cancellations.

Furthermore, intuitively, if boundary of a 4-manifold is non-empty and connected, then the boundary is a 3-manifold. As we will show in the coming sections, Kirby diagram is closely relatived to the surgery of 3-manifolds.

3.1 Review on Handle Decompositions

Handle decomposition is a powerful tool to study manifolds, we will have a brief review on that, further details can be found in [Y] by Yukio Matsumoto or other reference books on Morse theory.

3.1.1 Constructions

In this section, we let M be an smooth compact n-manifold without boundary. We try to construct a special kind of function on M called Morse function and see how it help us to perform handle decomposition.

Definition 3.1.1. A function $f : M \longrightarrow \mathbb{R}$ is said to be a Morse function if every critical point of f is nondegenerate, i.e. determinant of Hessian matrix at every critial point p of f is nonzero.

$$\det H_f(p) = \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right| \neq 0$$

with respect to the local coordinate around p.

In order to say a critial point is nondegenerate, we need to ensure it would not depend on the choice of local coordinate which is easy to be checked.

Theorem 3.1.2. *(Existence of Morse functions) Let $g : M \longrightarrow \mathbb{R}$ be a smooth function defined on M, then there exist a Morse function $f : M \longrightarrow \mathbb{R}$ arbitrarily close to g.*

This theorem not just states the existence of Morse function, it somehow states that the collection of Morse functions is dense in the set of smooth function on M.

Note that for a critial point p of f, we have $\frac{\partial f}{\partial x_i}(p) = 0$ for all $1 \leq i \leq n$ with respect to local coordinate $(x_1, \cdots, x_n)$. By the way, since $H_f(p)$ is symmetric and determinant of $H_f(p)$ is nonzero, it can be diagonalized with k positive and $n - k$ negative eigenvalues. Transition matrix for diagonalization combining with a suitable rescaling matrix gives us a matrix of changing of basis and therefore f can be expressed in the form of

$$f = f(p) - x_1^2 - x_2^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2$$

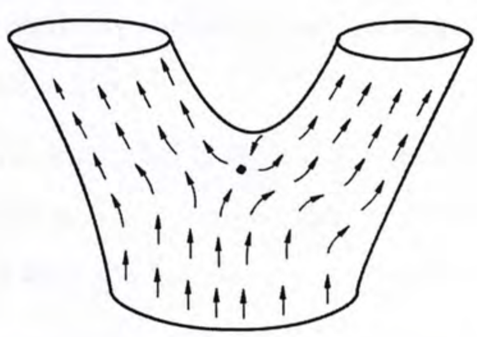
around a critical point. We also define an index for a critical point p equals to k which is the number of negative eigenvalues. Again, the definition of such index is independent from the choice of coordinates.

Definition 3.1.3. A vector field X on M is said to be a gradient-like vector field for a Morse function $f : M \longrightarrow \mathbb{R}$ if the following two conditions hold:

1. $X \cdot f > 0$ away from the critical points of f, i.e. X points in the directions which f is increasing.

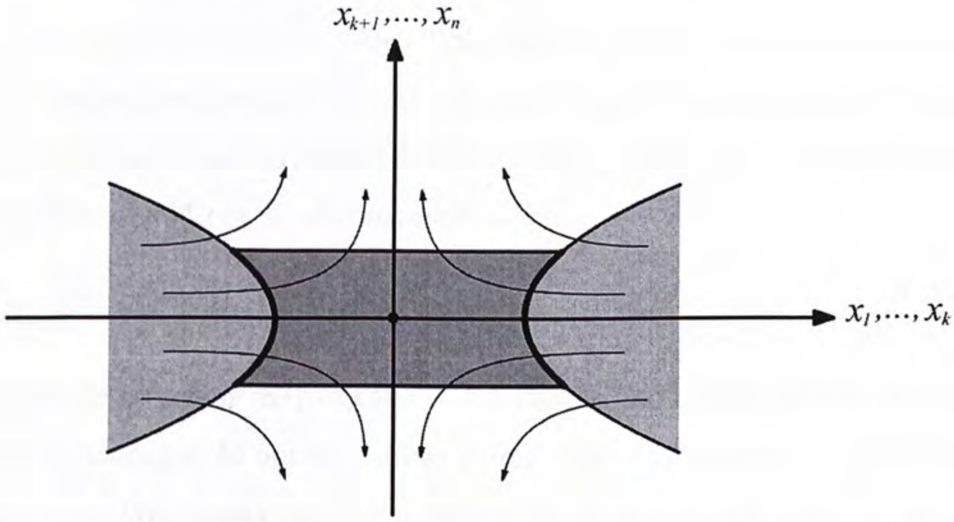
2. Around a critical point p, with a suitable choice of local coordinates, X can be written as its gradient vector field:
$$X = -2x_1\frac{\partial}{\partial x_1} - \cdots - 2x_k\frac{\partial}{\partial x_k} + 2x_{k+1}\frac{\partial}{\partial x_{k+1}} + \cdots + 2x_n\frac{\partial}{\partial x_n}$$
i.e. around every critical point p of f, X coincide with the gradient vector field.

Image f as a height function on M, we have the picture with the dot as a critical point.



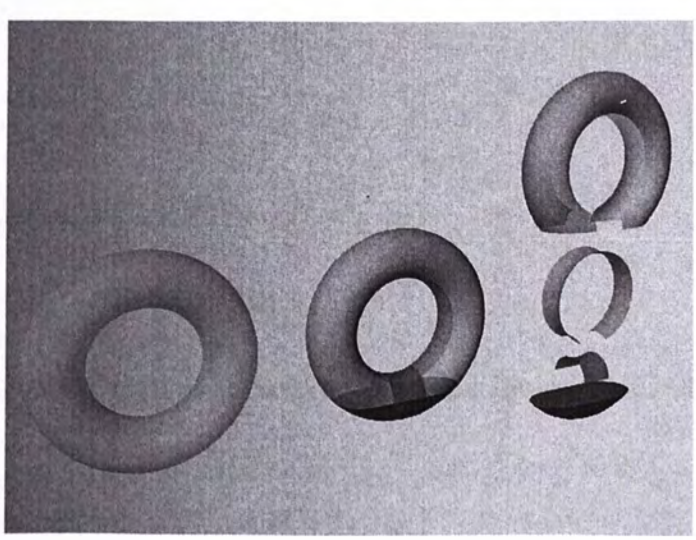
Theorem 3.1.4. *Suppose that $f : M \longrightarrow \mathbb{R}$ is a Morse function on M, then there exists a gradient-like vector field X for f.*

We jump over the technical details and understand how a gradient-like vector field helps us to decompose the manifold. Now suppose we have a smooth n-manifold M and a gradient-like vector. Then at each critical point p with index k, we attach $D^n = D^k \times D^{n-k}$, called k-handle ($(0,0) \in D^k \times D^{n-k}$ is attached to p) such that the vector field is compatible with the handles in the sense that the vector field is flowing-in on $\partial D^k \times D^{n-k} = S^{k-1} \times D^{n-k}$ and flowing-out on $D^k \times \partial D^{n-k} = D^k \times S^{n-k-1}$ like the figure below:



It follows that the manifold can be described by handles with indices and the way of attaching (attaching maps). By smoothing out corners, the attaching maps can be chosen to be smooth maps.

Example 3.1.5. A torus is embedded in $\mathbb{R}^3$ and choose the height function, prove the vector field of the height function is a gradient-like vector field and use it to decompose the torus into handles, we have the following:



On the right hand side, the top piece is a 2-handle, the two pieces in the middle are 1-handles and the bottom piece is a 0-handle.

Now we come to explain some terminologies. As the above, a disc D^n is realized as $D^k \times D^{n-k}$, we call it a k-handle. Since the manifold M is compact and connected, with suitable arrangements, we may assume it has only one 0-handle and one n-handle, also $(k+1)$-handles are attached after k-handles (it will be explained in next subsection). Then the manifold M can be decomposed as

$$M = (D^0 \times D^n) \cup_{\phi_{11}} (D_1^1 \times D_1^{n-1}) \cup \cdots \cup_{\phi_{(n-1)j}} (D_j^{n-1} \times D_j^1) \cup_{\phi_n} (D^n \times D^0)$$

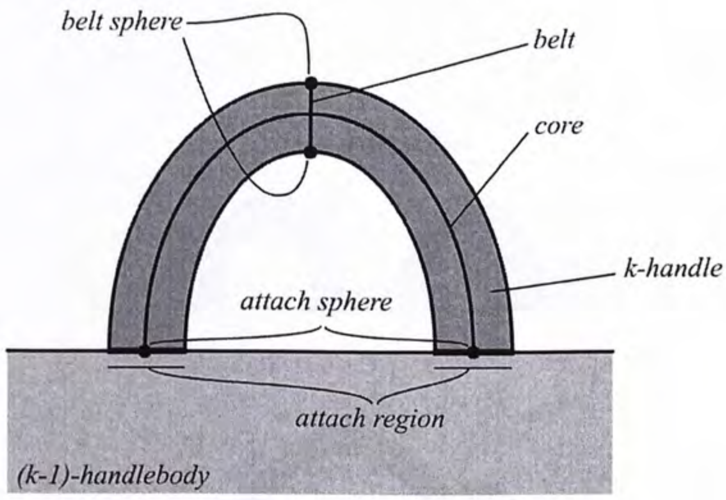
where ϕ_{ki} is the attaching map of the $i-th$ k-handle and it maps $\partial D_i^k \times D_i^{n-k}$ onto the boundary of the manifold obtained from gluing previous handles. It is called a handle decomposition. We usually use the notation M_k to denote the union of all i-handles where $1 \leq i \leq k$ and call it a k-handlebody, then we get

$$M_k = M_{k-1} \cup \{ \text{ } k\text{-handles } \}$$

Also we try to introduce these terms in order to make the discussion convenient: For a k-handle,

- core: $D^k \times \{0\}$
- belt: $\{0\} \times D^{n-k}$
- attaching region: $\partial D^k \times D^{n-k} = S^{k-1} \times D^{n-k}$
- attaching sphere: $\partial D^k \times \{0\} = S^{k-1} \times \{0\}$
- belt sphere: $\{0\} \times \partial D^{n-k} = \{0\} \times S^{n-k-1}$

There is an imagined picture to help memorizing the relations of them.



To conclude the above discussion, we have the theorem:

Theorem 3.1.6. *Any smooth compact manifold without boundary admits a handle decomposition.*

Proof. For a smooth compact manifold M without boundary, we can construct a Morse function f on M by theorem (3.1.2) and a gradient-like vector field X corresponding to f by theorem (3.1.4). Therefore, the manifold can be decomposed into handles as described before and such a decomposition is a handle decomposition. $\square$

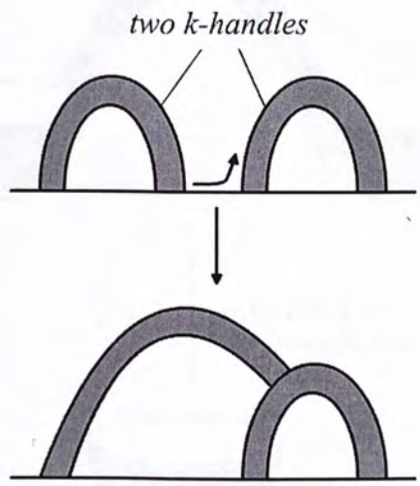
3.1.2 Handle Slides and Cancellations

The most important question to ask after introducing handle decomposition is:

If we choose different Morse functions and get different handle decompositions, how are those decompositions related?

It turns out that those decompositions are related by handle slides and cancellations which we will cover. As a review, we will only go through the facts, further details can be found in the reference text [Y] mentioned in the beginning of this section.

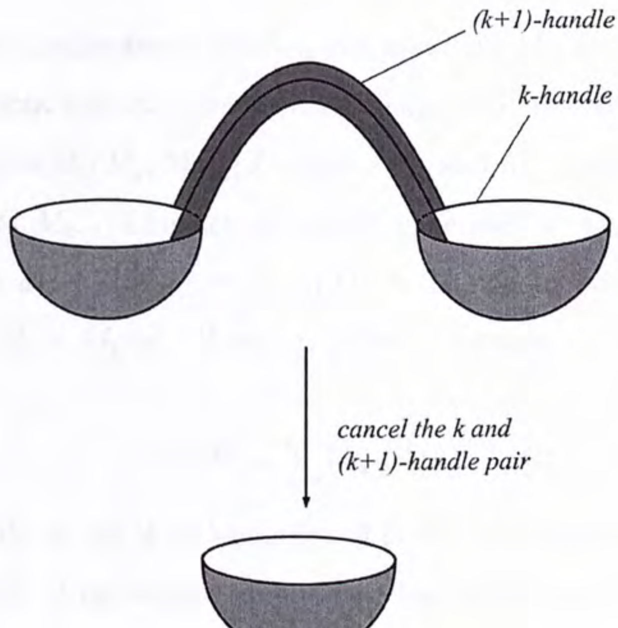
Handle slides



Theorem 3.1.7. *Suppose that a k-handle $D^k \times D^{n-k}$ is attached by an attaching map $\phi : \partial D^k \times D^{n-K} \longrightarrow \partial N$ on an n-dimensional manifold N with boundary, and suppose h_t is an isotopy of ∂N with $h_0 = id$ and $h_1 = h$. Then the new manifold obtained by using new attaching map $h \circ \phi$, $N \cup_{h\circ\phi} D^k \times D^{n-k}$, is diffeomorphic to the original one, $N \cup_\phi D^k \times D^{n-k}$.*

By using handle slides, we are able to slides handles such that the handles are attached in ascending order of indices, that means one can attach all k-handles and then $(k+1)$-handles to construct back the manifold, so we have the gluing of handles as discussed in last subsection, refer to lemma (3.33) in [Y].

Handle Cancellations



Theorem 3.1.8. *Suppose that a manifold N_1 is obtained from an n-dimensional manifold with boundary by attaching a k-handle, and N_2 is obtained from N_1 by attaching a $(k+1)$-handle:*

$$\begin{aligned} N_1 &= N \cup_\phi D^k \times D^{n-k} \\ N_2 &= N_1 \cup_\psi D^{k+1} \times D^{n-k-1} \end{aligned}$$

If the belt sphere $0 \times \partial D^{n-k}$ of the k-handle intersects the attaching sphere $D^{k+1} \times 0$ of the $(k+1)$-handle transversely at only a single point in the boundary ∂N_1, then N_2 is diffeomorphic to N.

To get the answer of the question at the beginning of this subsection, we have:

Theorem 3.1.9. *Given two handle decompositions for a manifold M, it is possible to get one to another by a sequence of handle slides, creations, cancellations.*

3.1.3 Calculation of Homology Groups

As mentioned at the very beginning, handle decomposition can help us to calculate the homology groups. We will explain it briefly here and reserve for later use.

We start from a handle decomposition of a manifold M (with arrangement so that those handles are attached with increasing index) and define the group of relative k-chains to be $C_k(M) = H_k(M_k, M_{k-1}; \mathbb{Z})$, where M_k and M_{k-1} represents k and $(k-1)$-handlebody. $H_k(M_k, M_{k-1}; \mathbb{Z})$ is actually freely generated by the k-handles. Then the boundary operator $\partial : C_k(M) \longrightarrow C_{k-1}(M)$ is defined by the long exact sequence of the triple (M_k, M_{k-1}, M_{k-2}). However, if M is oriented, it can be described by a simpler formula:

$$\partial h = \sum (B_i \cdot A) h_i$$

where h is a k-handle, h_i are $(k-1)$-handles, A is the attach sphere of h and B_i are belt spheres of h_i and $B_i \cdot A$ represents the intersection number of B_i and A on boundary ∂M_{k-1}. It can be proved that $\partial^2 = 0$ and so homology groups of that chain can be defined and it is exactly $H_k(M; \mathbb{Z})$. The proofs of the above can be found in chapter 4 of [Y].

3.2 Kirby Diagrams

A Kirby Diagram is a description of a 4-manifold by drawing its handles in $\mathbb{R}^3$. To simplify our discussion, we only focus on compact, oriented 4-manifolds without boundary. The reference book [GS] provides a more general picture of this topic and a lot of applications are covered.

3.2.1 Constructions

Recalling that a 4-manifold M can be decomposed into 0 to 4-handles. By duality, we know that the number of 0-handles and 4-handles must be equal, so as the number of 1-handles and 3-handles. Furthermore, since we only consider connected manifolds, we may assume that there is only one 0-handle (achieved by handle slides, creations and cancellations). Hence we start to construct back the manifold from a single 0-handle

$\{*\} \times D^4 = D^4$.

Instead of recording the whole handles, in Kirby diagrams, we only concern how the handles attach. However, the region that allow following handles to be attached is exactly a boundary of a 4-manifold, which is a 3-manifold, therefore we will be able to visualize the processes in $\mathbb{R}^3$.

The construction is started from a 0-handle. The region being attached by other handles is $\{0\} \times \partial D^4 = S^3$ which can be realized as $\mathbb{R}^3 \cup \{*\}$, by preturbation, we may assume that 1-handles and 2-handles are attached away from that point, therefore we can now try to draw how they attach in $\mathbb{R}^3$.

If we further investigate handle attachment in details, we discover that our diagrams need to include an important information, called framing.

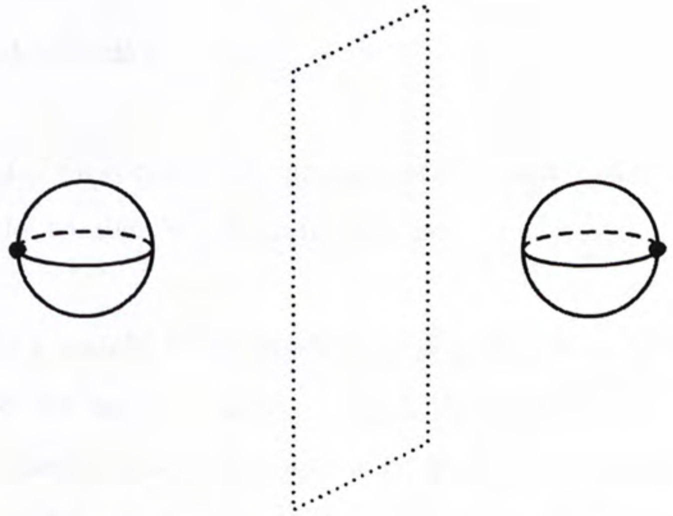
Framing

Formally, a k-handle $D^k \times D^{n-k}$ attaching to the boundary of an n-manifold ∂N can be described as a diffeomorphism $\phi : \partial D^k \times D^{n-k} \longrightarrow \partial D^k \times D^{n-k} \subseteq \partial N$. Therefore, counting ways to attach a k-handle to ∂N is just same as counting diffeomorphisms on $\partial D^k \times D^{n-k}$ (concerning the diffeomorphism group). As a matter of fact, the diffeomorphism group of $\partial D^k \times D^{n-k}$ is isomorphic to $\pi_{k-1}(O(n-k))$, the elements in this group are called framings. In the case of 4-manifolds, we will discover it mainly affects the attachment of 2-handles.

Attaching 1-handles

The attaching region of a 1-handle is $\partial D^1 \times D^3 = S^0 \times D^3 = D^3 \coprod D^3$ which is two disjoint 3-balls. Also the group $\pi_0(O(3)) = \mathbb{Z}_2$, thus there is a unique way to attach the 1-handle up to orientation preserving diffeomorphism, we only need to draw a pair of 3-balls in our diagram.

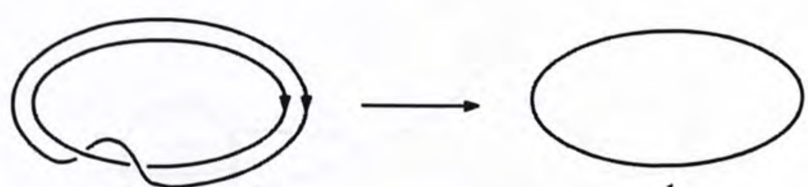
When we consider a path on that 1-handle $(t, *) \in D^1 \times \partial D^3 = D^1 \times S^2$ where $t \in [0, 1]$ and $*$ is a fixed point in S^2. Then the path starts at a point in the left 3-ball in the diagram, then it goes over the 1-handle and seems disappeared in the diagram and finally reappears in the right 3-ball again, so we only see two points in the diagram as following.



Keep in mind that the manifold is oriented, for such a path mentioned above, the starting point and the end point should appear as a pair of symmetrical points in the diagram (symmetric about the plane in our figure).

Attaching 2-handles

The attaching region of a 2-handle is $\partial D^2 \times D^2 = S^1 \times D^2$ which is a solid torus. To simplify the diagram, we only draw the image of $S^1 \times 0$ under the attaching map, which is a knot in $\mathbb{R}^3 \subseteq S^3$. However the group $\pi_1(O(2)) \cong \mathbb{Z}$, if we only record the knot, the information of framing will lose. To compensate the loss, we need to put an integer there. There is a canonical choice of that integer which is the linking number (see the definition in section(5.4)) of the image of a longitude of the attach region (solid torus) and $S^1 \times 0$ of the solid torus, refer to chapter 4 of [GS].

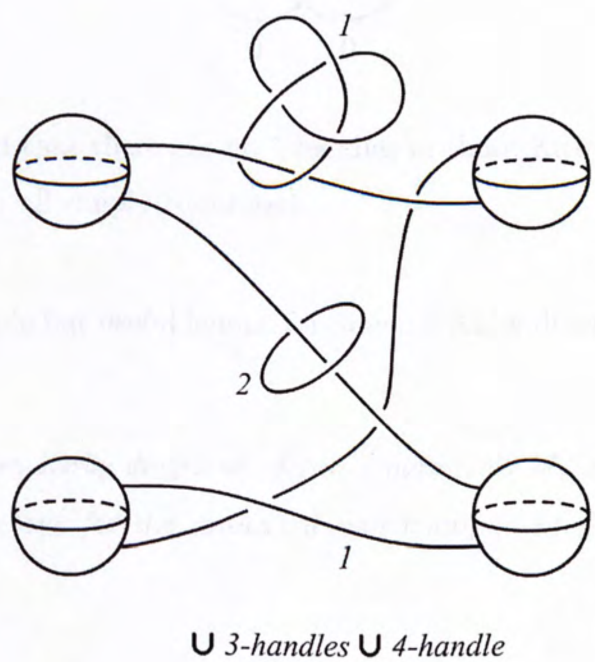


Attaching 3 and 4-handles

Similar to before, we consider the group $\pi_2(O(1))$ and $\pi_3(O(0))$. They are both trivial groups, so the trouble from framing does not come up.

Boundary of the manifold M we considering is empty, then the boundary of union of all 3-handles and the single 4-handle is exactly the boundary of union of 0, 1 and 2-handles, i.e. ∂M_2 where M_2 is the 2-handlebody. Furthermore, union of 3 handles and the single 4-handle is diffeomorphic to $\#m(S^1 \times D^3)$ where it is the connected sum of m copies of $S^1 \times D^3$ and m is the number of 3-handles. As before, asking how to glue back $\#m(S^1 \times D^3)$ is just counting the diffeomorphism of $\partial(\#m(S^1 \times D^3)) = \#m(S^1 \times S^2)$ which is unique up to orientation. Also any self-diffeomorphism of $\#m(S^1 \times S^3)$ can be extended uniquely over $\#m(S^1 \times D^3)$. Therefore, it is not required to keep track on the 3 and 4-handles if ∂M is empty.

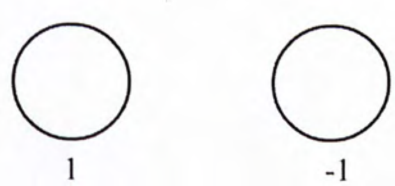
Here is an example of completed Kirby diagram:



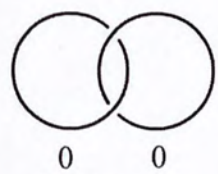
If there is no confusion, we will drop the tag denoting 3-handles and the 4-handle. However, sometimes when we focus on 4-manifolds with boundaries, it would be meaningful to keep that.

Also we would like to give the Kirby diagrams of some simple 4-manfolds.

Example 3.2.1. $\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2$ is given by unknotted circles with 1 and -1 as framings respectively.



Example 3.2.2. $S^2 \times S^2$ is given by the Hopf link with each has framing 0.



It is not surprised that there are no 1-handles in those Kirby diagrams of our examples since they are all simply connected.

It comes to a simple but useful lemma for drawing Kirby diagram of connected sum of 4-manifolds.

Lemma 3.2.3. *Given Kirby diagrams of two 4-manifolds M_1 and M_2, the combined Kirby diagram is the one for the connected sum manifold $M_1 \# M_2$ (i.e. drawing all handles together).*

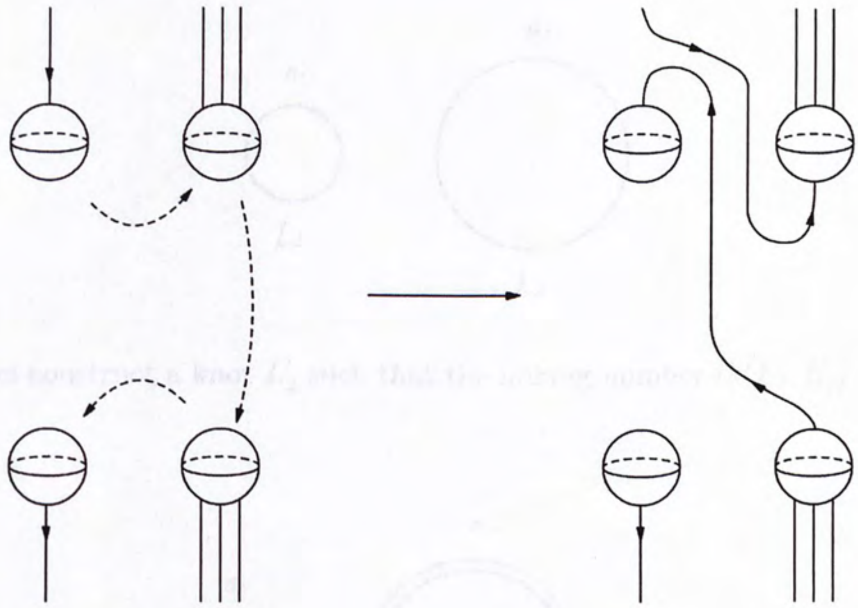
3.2.2 Handle Slides and Cancellations

As we knew before, a Kirby diagram is just recording a handle decomposition of a 4-manifold. Naturally, we would like to ask how handle slides and cancellations affect a Kirby diagram.

Handle Slides

1-handle

Handle Sliding for a 1-handles can be easily described as the following figure.

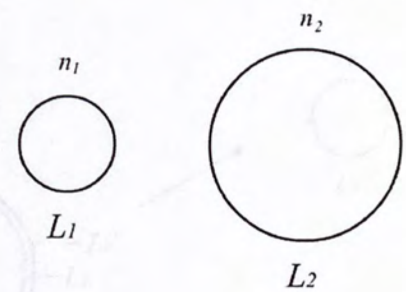


It just like moving one of 3-balls of the attaching region of a 1-handle, passing another 1-handle and come back the boundary of the 0-handle.

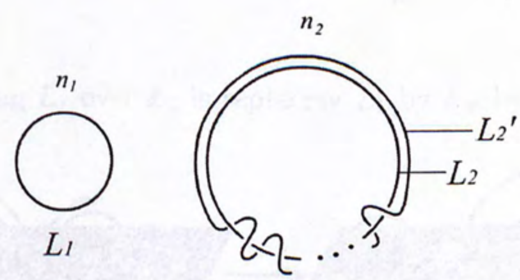
2-handle

Handle Sliding for a 2-handle is a little bit harder. Basically, it is required to determine the shape after a 2-handle slides over another 2 handle (that is a new knot in $\mathbb{R}^3 \subseteq S^3$), and also the framing of it. We first give the way to perform that and explain why it is true after.

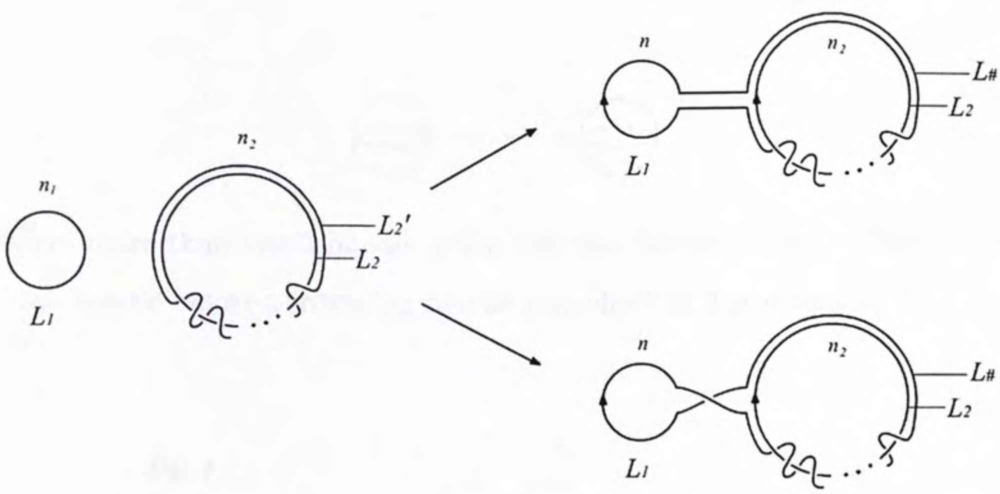
Suppose two 2-handles L_1 and L_2 are given in a Kirby diagram as following, they appear as two knot with an integer for each knot to denote the framing, here are n_1 and n_2. We want to slide L_1 over L_2.



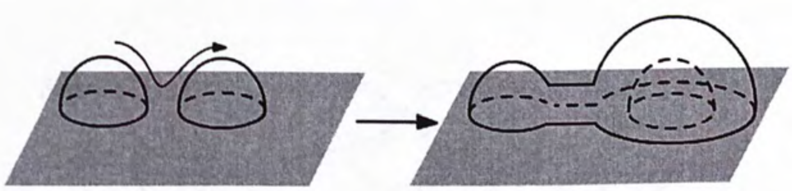
First, we construct a knot L_2' such that the linking number $lk(L_2, L_2') = n_2$.



Then we can form a new knot $L_\# = L_1 \#_b L_2'$ where b is a band connecting L_1 and L_2' and $\#_b$ denote the connected sum of L_1 and L_2' by using b. After sliding L_1 over L_2, all other handles are unchanged, including framings of 2-handles, in the diagram except L_1 is replaced by $L_\#$ and the framing of $L_\#$ is given by $n = n_1 + n_2 + 2lk(L_1, L_2)$. In order to calculate $lk(L_1, L_2)$, it requires us to determine the orientations of L_1 and L_2 first and it actually depends on the choice of the band as shown below.



It is clear why sliding L_1 over L_2 is replacing L_1 by $L_\#$ by the following figure.



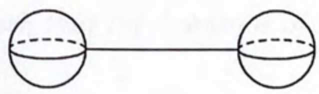
Also the new framing can be obtained by the following calculation:

$$\begin{aligned}
\text{framing of } L_\# &= lk(L_\#, L_\#) \\
&= lk(L_1 \#_b L_2', L_1 \#_b L_2') \\
&= lk(L_1, L_1) + lk(L_2', L_2') + 2lk(L_1, L_2') \\
&= n_1 + n_2 + 2lk(L_1, L_2) \qquad \text{(note that } lk(L_1, L_2') = lk(L_1, L_2))
\end{aligned}$$

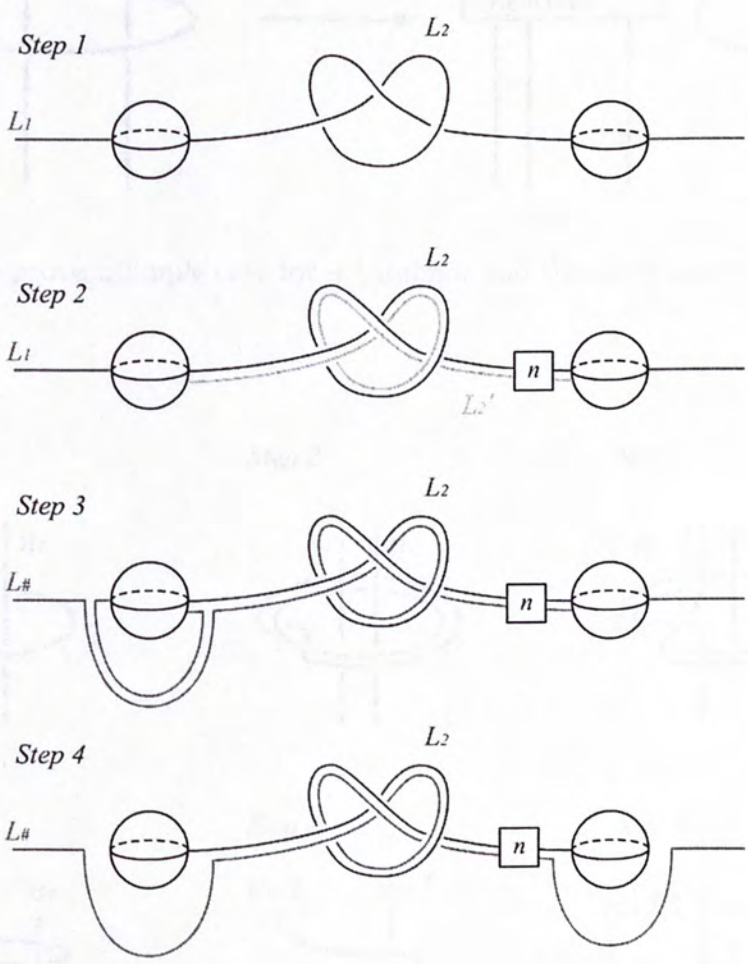
Handle Cancellations

1-2 handle pair

Recall that theorem (3.1.8) tell us if belt sphere of a 1-handle intersect attaching sphere of a 2-handle, then we can cancel that pair of handles. It is exactly the case of the following figure.



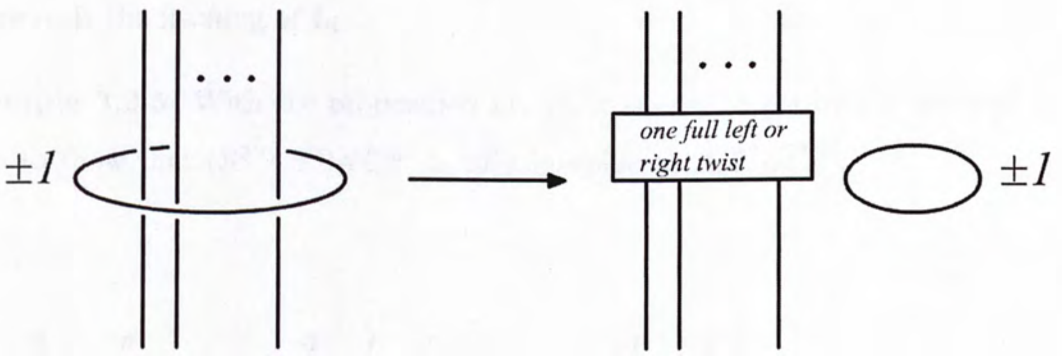
If there are more than one 2-handle going over the target 1-handle, then we have to slide those handle before performing handle cancellation. For example,



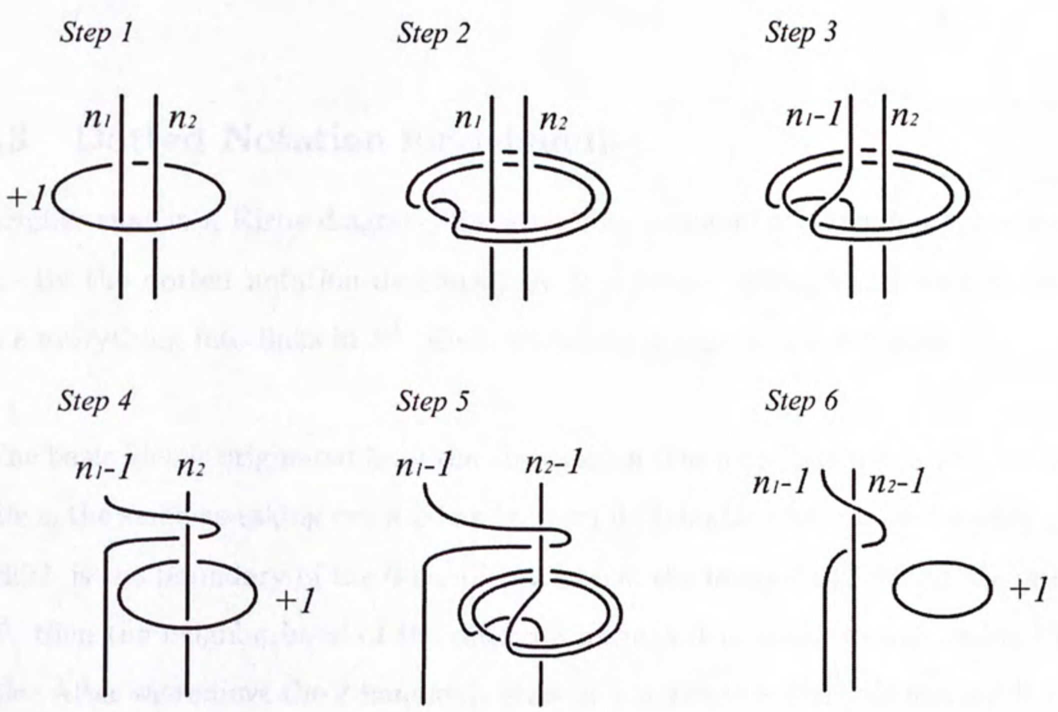
L_2 and L_2' are twisted n-times for a suitable n so that $lk(L_2, L_2') = n_2$. Finally, just take out the L_2 and the two 3-balls (blue curves and black balls), then we finish the cancellation.

We finish this subsection by several examples.

Proposition 3.2.4. *An unknot with framing* ± 1 *can be moved away from the rest of the link as the following diagram such that the framings of the other arc are changed by* ∓ 1.



Proof. We only prove a simple case for +1 unknot and the other cases are just similar.

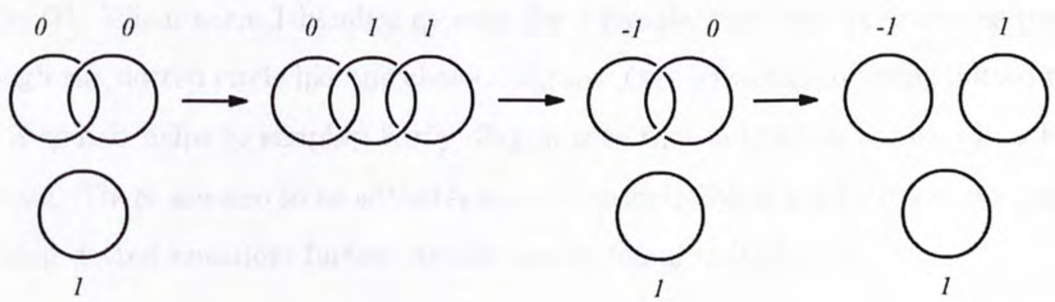


The framing of an arc after pulling out the +1 unknot

$$\begin{aligned} &= n_i + 1 + 2lk(+1 \text{ unknot}, L_i) \\ &= n_i + 1 - 2 \\ &= n_i - 1 \end{aligned}$$

where n_i is the framing of L_i. □

Example 3.2.5. With the proposition above, it is easy to perform a series of operations to show that $(S^2 \times S^2)\#\mathbb{CP}^2$ is diffeomorphic to $2\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$.

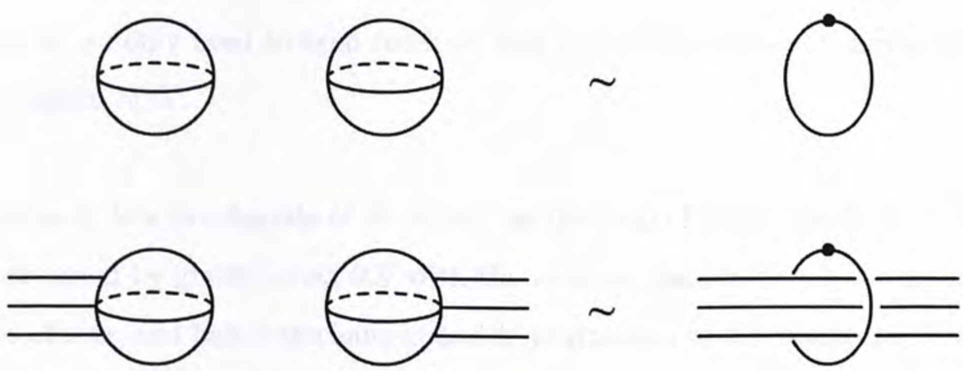


3.2.3 Dotted Notation for 1-handles

Remember that in a Kirby diagram, the attaching region of a 1-handle appear as two balls. By the dotted notation developed by S. Akbulut [A1], [AK1], we are able to reduce everything into links in S^3. Here, we briefly go through it for later use.

The basic idea is originated from the observation that attaching a 1-handle to the 0-handle is the same as taking out a 2-handle from it. Imagine that we have a disk D^2 in S^3 which is the boundary of the 0-handle. We push the interior of D^2 into the interior of D^4, then the neighborhood of the disk can be regarded as a 2-handle inside the 0-handle. After we remove the 2-handle, it gives us a union of a 0-handle and a 2-handle.

We draw a circle with a dot to represent what we do and we have the following diagram.



The dotted circle can be regarded as the boundary of the disk whose interior is inside D^4. When some 1-handles go over the 1-handle, they can be drawn as passing through the dotted circle like the above diagram. One advantage of using dotted notation is that it helps to simplify Kirby diagrams so that only knots appears in a Kirby diagram. There are also some advantages, such as simplifying some surgery operations, by using dotted notation, further details can be found in [GS].

Then we try to come across an interesting example, which is also very useful for us later.

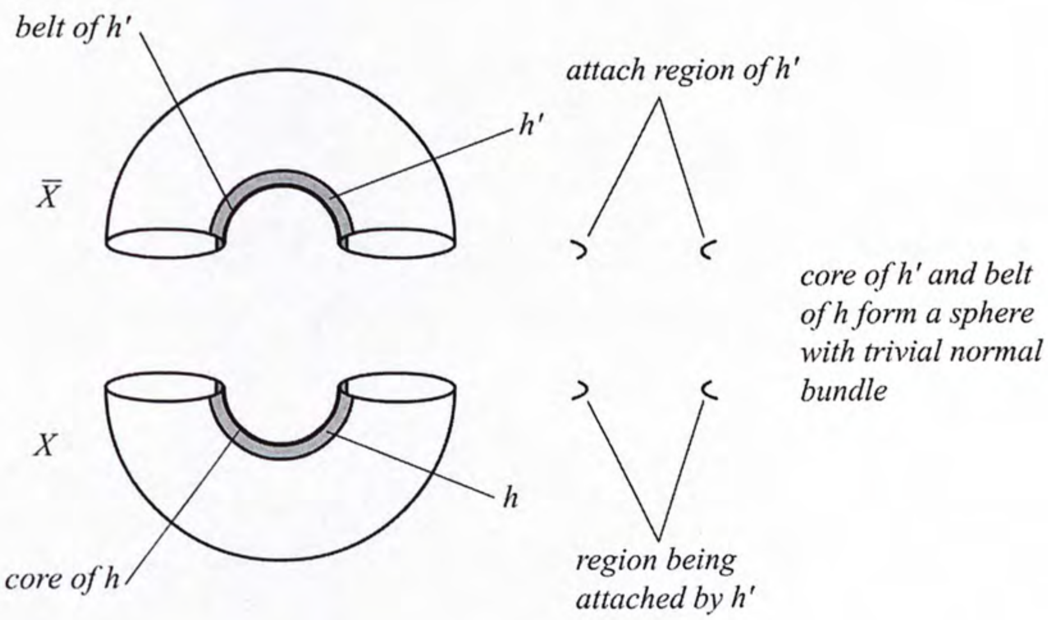
Example 3.2.6 (Doubles). Suppose X is a compact n-manifold with boundary (in particular $n = 4$), we define the double of X to be

$$DX = \partial(I \times X) = X \cup_{id_{\partial X}} \overline{X}$$

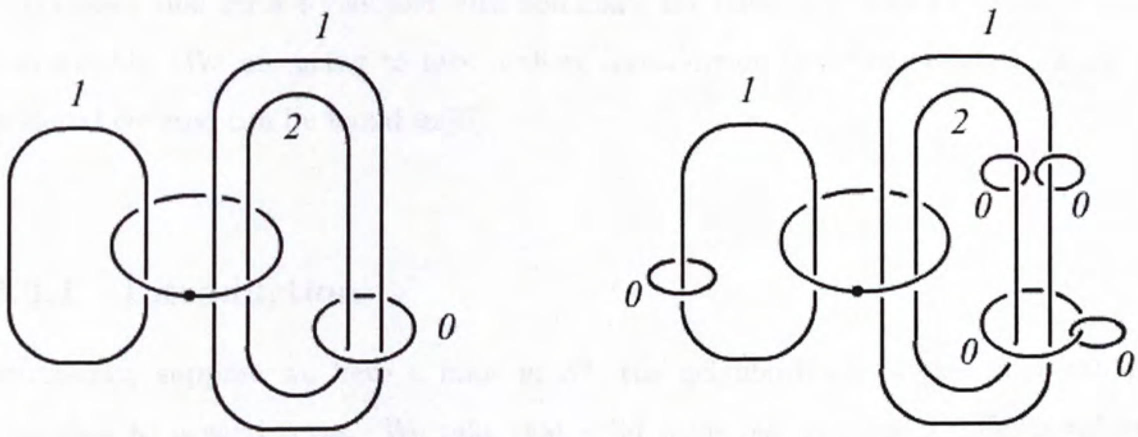
where $\overline{X}$ is the manifold same as X with reversed orientation. Imagine we take two copies of the same manifold X, they have the same boundary and so we can glue them together and form the double DX. It is obvious that the boundary of DX is empty.

Now given a 4-manifold X with boundary and its Kirby diagram, we would like to construct the Kirby diagram of the double DX. $\overline{X}$ is just the same manifold as X, by duality we can regard $\overline{X}$ as union of 2, 3 and 4-handles and adding them to the Kirby diagram of X to construct DX, also the number of 2, 3 and 4-handles of $\overline{X}$ equal to the number to 2, 1 and 0-handles of X respectively. However we know $\partial(DX)$ is empty, so we only need to keep track on how those 2-handles of $\overline{X}$ are added to the Kirby diagram of X.

Suppose h' is a two handle of $\overline{X}$ which the the dual of a two handle h of X. Since DX is obtained by gluing along ∂X with the identity map, h' is just a copy of h with the roles of core and belt interchanged and h' is attached to ∂X along the belt sphere. Furthermore, the core of h' and belt of h form a sphere with trivial normal bundle, hence the neighborhood of the sphere is $S^2 \times D^2 = (D^2 \cup D^2) \times D^2$ and the framing of the attaching region of h' is 0.

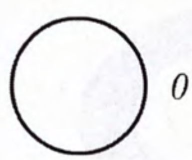


Therefore, to conclude the results above, when one wants to construct the Kirby diagram of DX from X, he only needs to add a 0-framed circle to each link component representing a 2-handle in X.

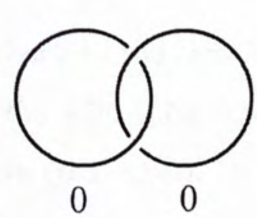


X $\qquad$ DX

Example 3.2.7. For the 4-manifold has only one 0-handle and one 1-handle with the following Kirby diagram



It is actually $S^2 \times D^2$ with boundary $S^2 \times S^1$. Now if we form the double of it, according to the above, then the double will have the Kirby diagram which is $S^2 \times S^2$



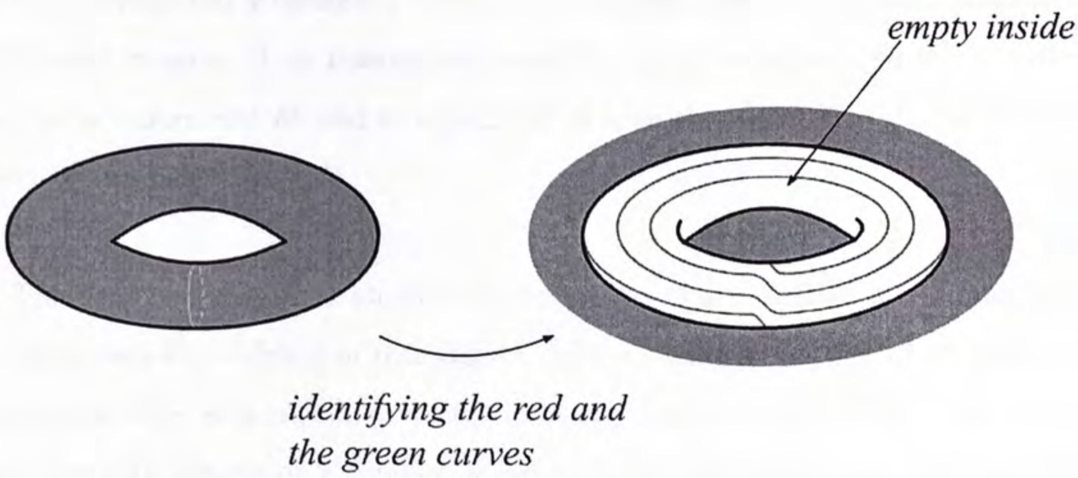
Actually, if we have two copies of $S^2 \times D^2$ and glue along the same boundary, note that the two disks in the second factor are glued along the boundary circles and becomes S^2. Therefore, the double is really $S^2 \times S^2$.

3.3 3-Manifolds: As Boundaries of 4-Manifolds

It is known that for a 4-manifold with boundary, the boundary must be union of some 3-manifolds. We are going to give a short introduction how they related. Most of material covered can be found in [S].

3.3.1 Introduction

Intuitively, suppose we have a knot in S^3, the neighborhood of this knot can be identified as a solid torus. We take that solid torus out and use a diffeomorphism $\phi : S^1 \times S^1 \longrightarrow S^1 \times S^1$ to glue back the solid torus to the remaining part to form a new 3-manifold.



The solid torus is then glued back to the remaining part of that S^3 (gery region) such that the image meridian of the solid torus (indicated by the green circle) under the gluing map is the twisted curve (red curve).

This famous operation is called Dehn surgery. In fact, the resulting manifold is completely determined by how a meridian is glued back which can be described by two integers. To be more explicit, the image of the a meridian under the gluing map must isotopic to a curve of the form $c = p \cdot m + q \cdot l$ where p, q are integers and m and l denote the the meridian and longitude respectively, then the surgery can be described by a pair of integers (p, q). Fortunately, it was found that if $p/q = s/t$, then the surgeries corresponding to the pair (p, q) and (s, t) are simply the same one, therefore it is reasonable to denote this kind of surgery by a reduced fraction and sometime we call it a rational surgery. Furthermore, if $q = \pm 1$, we call it an integral surgery. It can be summarized by the following theorem.

Theorem 3.3.1. *Every closed orientable 3-orientable manifold can be obtained from S^3 by an integral surgery on a link $L \subseteq S^3$.*

Hence, we know that every closed orientable 3-manifold can be described by a diagram consisting a link in S^3 (by preturbation, it is in $\mathbb{R}^3$) and an integer denoting the framing coefficient for each component. On the other hand, for a Kirby diagram for a simply connected 4-manifold, there is no 1-handle, thus it is again a diagram with a link and integers. It is reasonable to ask the relation between them. It turns out that for a 4-manifold M and a 3-manifold N with the same diagram (same link and integers), we have $\partial M = N$.

This is an important result linking the studies of 4-manifolds and 3-manifolds. It can be shown that adding or deleting an unknot with framing ± 1 which does not intersect the other existing components in a Kirby diagram will not affect the boundary. Together with sliding of 2-handles, if we perform this operations to a diagram of a 3-manifold just like what we do to Kirby diagrams, the resulting 3-manifolds corresponding to the resulting diagram would be homeomorphic to the original one. We call those two kind of operations to diagrams for 3-manifolds Kirby moves. R. Kirby finally proved that:

Theorem 3.3.2. *The closed oriented 3-manifolds obtained by surgery on framed links L_1 and L_2 are homeomorphic by an orientation preserving homeomorphism if and only if the link L_2 can be obtained from L_1 by a sequence of Kirby moves.*

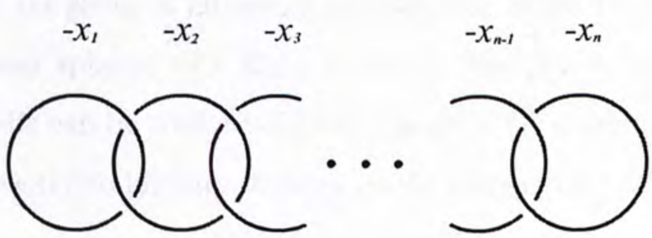
3.3.2 Lens spaces

One of the important kind of closed oriented 3-manifolds is Lens space which is performing a (p/q)-surgery along an unknotted circle in S^3, it is denoted by $L(p,q)$. As the construction before, some of the lens spaces may be homeomorphic with distinct pairs of integers. To conclude this, we have the following theorem.

Theorem 3.3.3. *The lens spaces $L(p,q)$ and $L(p,q')$ are homeomorphic if and only if $q = \pm q'^{\pm 1}$ (mod p).*

By theorem(3.3.1), to obtain a lens space $L(p,q)$, instead of performing a (p/q)-surgery, it can be obtained by an integral surgery along a link. Here, we need to emphasize that we may no longer performing surgery along a knot now. Here is the answer of the question that which kind of links we should surgery along and the framing coefficient:

Theorem 3.3.4. *Any lens space $L(p,q)$ can be obtained by surgery along the link as below*

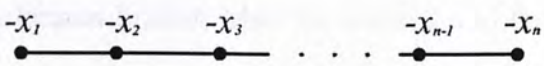


where $\frac{p}{q} = -[x_1, x_2, \cdots, x_n]$ is a continued fraction decomposition,

$$[x_1, x_2, \cdots, x_n] = x_1 - \cfrac{1}{x_2 - \frac{1}{\cdots - \frac{1}{x_n}}}$$

Proof. The details of the proof can be found in section(2.3) of [Sa], but the idea is basically showing by induction that a len space $L(p,q)$ can be obtained in the way described above if $\frac{p}{q}$ has a continued fraction decomposition, then using the key result that every fraction has a continued fraction decomposition. Hence each lens space $L(p, q)$ can be obtained. $\square$

To make the notation convenient, we simply use the tree to represent the unknotted circles as following:



For the later use, we make this theorem to conclude about the homology groups of lens spaces.

Theorem 3.3.5. *Let $L(p, q)$ be the lens space mentioned above. Then*

$$H_k(L(p,q)) = \begin{cases} \mathbb{Z} & \textit{if } k\textit{=0,3} \\ \mathbb{Z}_p & \textit{if } k\textit{=1} \\ 0 & \textit{otherwise} \end{cases}$$

3.4 Linear Plumbing

In this section, We are going to introduce an operation called linear plumbing. Given two disk bundles over spheres with Euler numbers. We pick a point on each sphere, then each disk bundle can be trivialized locally as $D^2 \times D^2$ around each point we have just picked. Then we try to identify those pieces by the map $\phi : D^2 \times D^2 \longrightarrow D^2 \times D^2$ defined by

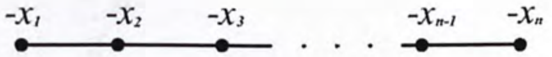
$$\phi(a, b) = (b, a)$$

After we smooth out the corners, the resulting solid can be interpret as a 4-manifold with boundary. The operation is called plumbing of two disk bundles.

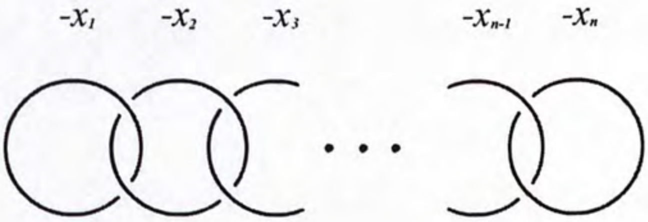
Of course, we can extend this operation to more than two disk bundles, also the sphere bundles may not be required to be linked one by one. How the sphere bundles plumbed can be represented by a tree, called plumbing tree. Each vertex represents a sphere bundle with an integer attached to each vertex to represent the Euler number of the corresponding bundle, every the sphere bundles represented by adjacent vertices are plumbed together. If a plumbing tree is linear, then the plumbing is called a linear plumbing.

Then we come to a lemma linking what we learned and linear plumbing.

Lemma 3.4.1. *The four manifold with boundary obtained by linear plumbing according the tree*



is exactly the one with the following Kirby diagram.



Furthermore, that four manifold can be realized as a neighborhood of spheres with self-intersection according the integers of those vertices and intersection number of a pair of adjacent spheres is 1.

Although we will not cover the proof here, actually it is again proved by induction on the number of unknots and at least it is easy to verify that it is true for only one unknot. A local trivialization of a sphere bundle as $D^2 \times D^2 = D^4$ can be regard as a 0-handle, the remaining part can be realized as a 2-handle and the Euler number of the bundle is exactly the number of times that the 2-handle twisted (i.e. the framing coefficient).

When we talked about lens spaces, we have graphs of links and trees with integers specified. However, when we mentioned about linear plumbing, we encountered the same kind of graphs of links and trees with integers as well. However, with the lemma above and results discussed last section, the relation between all of them are clear now. If we perform a linear plumbing according a plumbing tree with a set of integers, then the 4-manifold we obtained is the one with the Kirby diagram with a link consisting unknotted circles and the same set of integers as we have just seen. Also the boundary of that 4-manifold is a lens space represented by a linear tree with the same set of integers. How to obtain the lens space is performing integral surgery along the same link in S^3 with that set of integers.

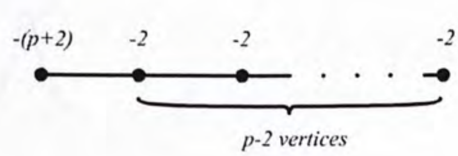
3.5 Rational Blowdown

After much preparing works, we come to a new technique invented by R. Fintushel and R. Stern [FS2] and it enables us to have a grant step forward on finding exotic structures of 4-manifolds.

Before going on, we first give the definition of rational homology ball.

Definition 3.5.1. If M is a n-dimensional manifold and $H_p(M;\mathbb{Q}) = H_p(D^n;\mathbb{Q})$ for all p, then M is called a rational homology n-ball.

Suppose we perform a linear plumbing according the following tree



we denote the resulting 4-manifold by C_p. As theorem(3.3.4), we know that the boundary of C_p is $L(p^2, 1-p)$. Also we can show that this special kind of lens space bounds a rational 4-homology ball, denoted by B_p, i.e. $\partial B_p = L(p^2, 1-p)$

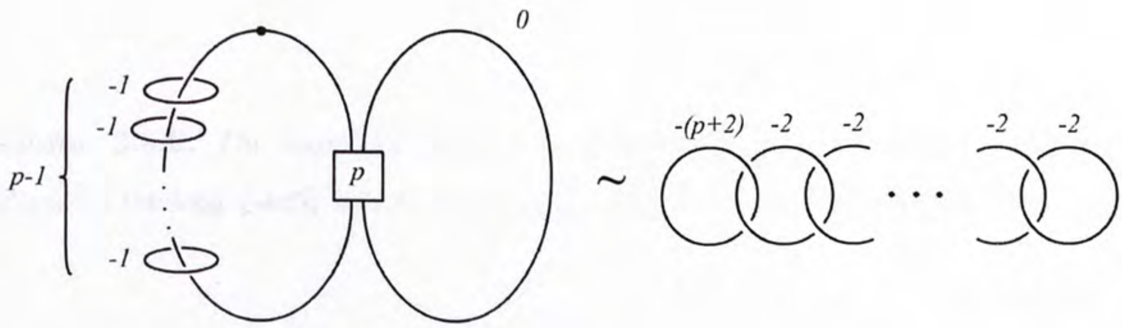
Now, if M is a 4-manifold and C_p is embedded in M, then we can replace C_p by B_p and construct a new manifold,

$$M' = (M - \text{int } C_p) \bigcup_{L(p^2,1-p)} B_p$$

This operation is called rational blowdown. A usual blowdown is replacing the neighborhood of a (-1)-sphere by a 4-ball, but now, we replace a neighborhood of a certain configuration of spheres by a rational homology 4-ball, so we have the name rational blowdown.

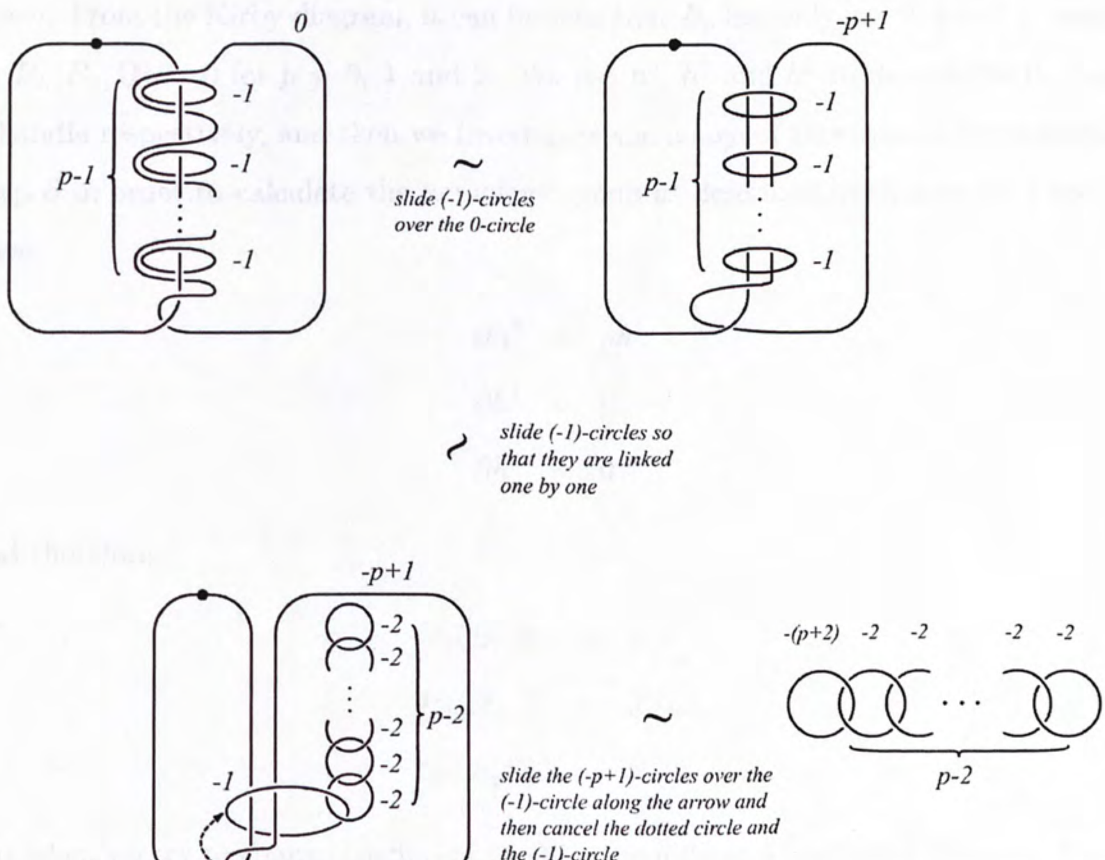
What we are going to do in the remaining part of this section is showing that $L(p^2, 1-p)$ really bounds a rational homology 4-ball. In order to make use what we have introduced, it will be proved by using Kirby Calculus. However, other approaches appears in [FS2] and [S] or some other texts.

Lemma 3.5.2. *The two manifolds represented by the following two Kirby diagrams are diffeomorphic.*



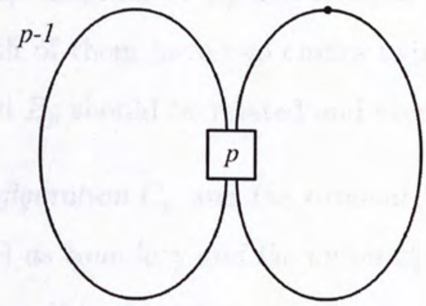
Recall that the right diagram is exactly the manifold C_p. Here, we have to emphasis that there are no 3 and 4-handles in both of the two Kirby diagrams.

Proof. We try to perform a sequence of handle sliding and handle cancellations to transform the right diagram to the left one.



□

Lemma 3.5.3. *The manifold denoted by B_p with the following Kirby diagram is a rational homology 4-ball, that is $H_k(B_p;\mathbb{Q}) = H_k(D^4;\mathbb{Q})$ for all integer $k \geq 0$.*



Proof. From the Kirby diagram, it can be seen that B_p has only one 0, 1 and 2-handle, so $H_k(B_p;\mathbb{Q}) = 0$ for $p \neq 0$, 1 and 2. We use h^0, h^1 and h^2 to denote the 0, 1 and 2-handle respectively, and then we investigate the image of them under the boundary map ∂ in order to calculate the homology group as described in chapter 3. Then we have,

$$\begin{aligned} \partial h^2 &= ph^1 \\ \partial h^1 &= 0 \\ \partial h^0 &= 0 \end{aligned}$$

and therefore,

$$\begin{aligned} H_2(B_p;\mathbb{Z}) &= 0 \\ H_1(B_p;\mathbb{Z}) &= \mathbb{Z}/p\mathbb{Z} \\ H_0(B_p;\mathbb{Z}) &= \mathbb{Z} \end{aligned}$$

but when we try to change coefficient to $\mathbb{Q}$ by the universal coefficient theorem. Recall that $H_*(M;\mathbb{Q}) = H_*(M;\mathbb{Z}) \otimes \mathbb{Q}$ and $\mathbb{Z}/p\mathbb{Z} \otimes \mathbb{Q} = 0$. Finally, we find that,

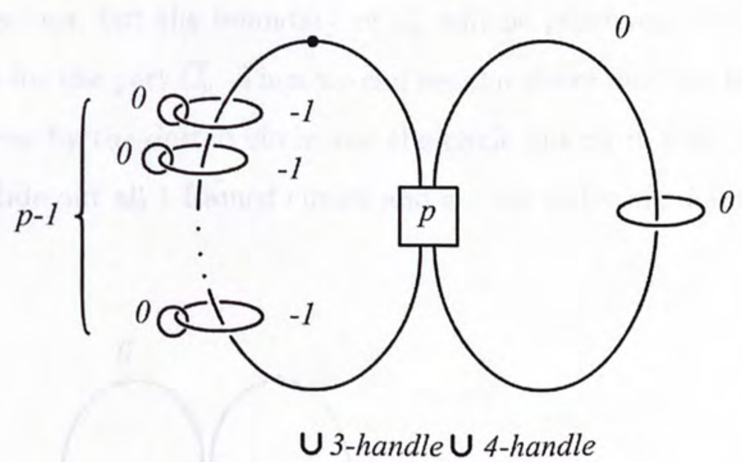
$$\begin{aligned} H_2(B_p;\mathbb{Q}) &= 0 \\ H_1(B_p;\mathbb{Q}) &= 0 \\ H_0(B_p;\mathbb{Q}) &= \mathbb{Q} \end{aligned}$$

Thus, $H_k(B_p;\mathbb{Q}) = H_k(D^4;\mathbb{Q})$ for all integer $k \geq 0$ and B_p is so a rational homology 4-ball. $\square$

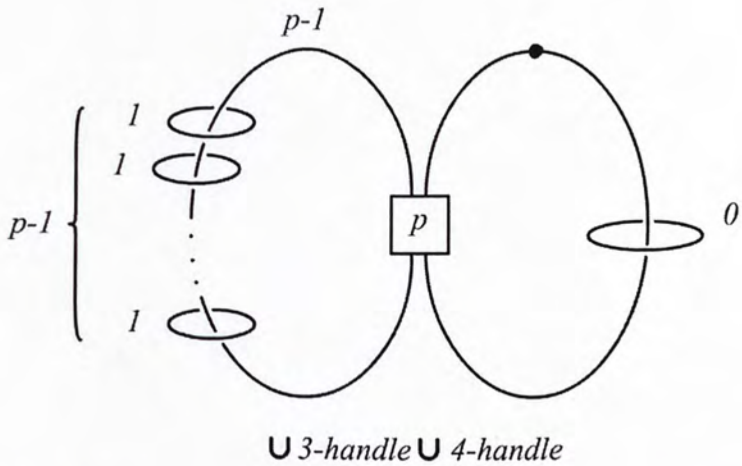
This lemma provides the diagram of B_p and it looks similar to the new diagram of C_p in lemma(3.5.2), both of them have two circles twisted together by p-times. It provides a clue that C_p and B_p should be related and eventually, we find that:

Theorem 3.5.4. *The configuration C_p and the rational homology 4-ball B_p have the same lens space $L(p^2, 1-p)$ as boundary and the union $C_p \cup \overline{B_p} = \#(p-1)\overline{\mathbb{CP}^2}$, where $\overline{B_p}$ denotes B_p with the reversing orientation.*

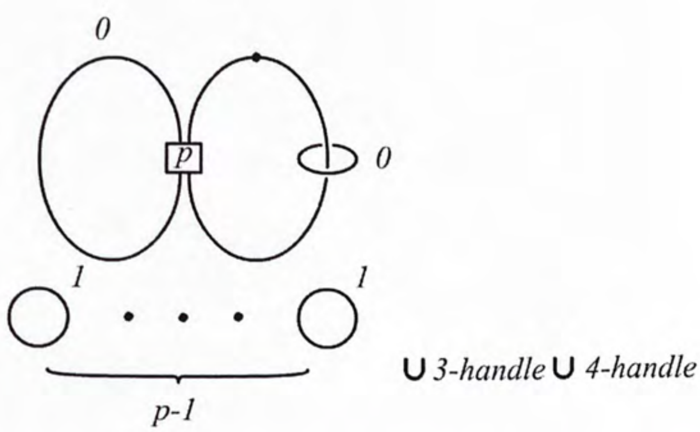
Proof. Firstly, we form a double DC_p of C_p, refer to example(3.2.6). There is only one 3-handle in DC_p beacuse there is only one 1-handle in C_p.



Then, we perform surgery twice inside $C_p \subseteq DC_p$, it corresponds to interchange the dot and the 0-framing of the two circles linking together by p-times. Also we blow down all (-1)-framed circles in C_p and we have the following diagram.



Recall that C_p will be changed to another 4-manifold after those surgeries and blowing down operations, but the boundary of C_p will be preserved, by the way, there is nothing change for the part $\overline{C_p}$. Then we can see the above diagram is exactly $B_p \cup \overline{C_p}$ where B_p is given by the dotted circle and the circle linking it with p-times. Furthermore, we can slide out all 1-framed circles and get the following diagram.



Then the 0-framed circle linking with the dotted circle 1-time can be cancelled together (handle cancellation pair) and the remaining 0-framed circle can be cancelled with the 3-handle. Finally, we have $p-1$ 1-framed circles which means $B_p \bigcup \overline{C_p} = \#(p-1)\mathbb{CP}^2$ and reversing the orientation, we get $C_p \bigcup \overline{B_p} = \#(p-1)\overline{\mathbb{CP}^2}$. $\square$

Part II

Examples of Exotic Structures

Chapter 4

$m\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$

4.1 Introduction

$m\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$ is a kind of complex surface which is relatively simply to study, since we know that $\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2$ admit metrics of positive scalar curvature, namely the Fubini-Study metric. The result of Gromov and Lawson (Theorem A in [GL]) implies that every manifold of the type

$$m\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$$

admits a metric of positive curvature. Furthermore we know the intersection form of $\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$ is $m\langle 1\rangle \oplus k\langle -1\rangle$ and that $b_2^+ = m$. Thus by theorem (1.3.2) and its extension to the case of $b_2^+ = 1$, all Seiberg-Witten invariants vanish.

One simple strategy to obtain an exotic structure is constructing a simply connected symplectic manifold Y. By calculating the intersection form and use the classifcation theorem (2.2.1) to show Y is homeomorphic to our target $m\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$. Finally, since Y is a symplectic manifold, it admits a canonical almost complex structure and by theorem(1.3.6), the Seiberg-Witten invariant for the corresponding *spin*^{c} structure is ± 1, then we get a manifold with distinct differentiable structures.

4.2 Example: $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$

In particular, we pick $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$ as an example to show the ideas and difficulties of the proof. This section is based on [P1] by Jongil Park, we will cover the idea of the constructions and discuss details as a supplementary.

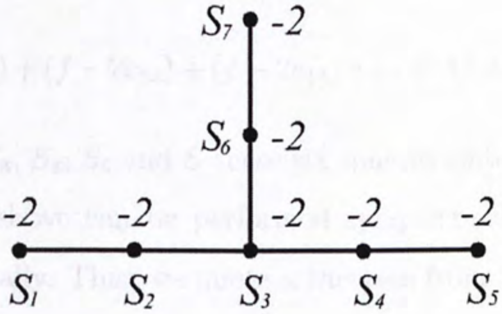
To make our notations clear, we have a short remark before going on. Cohomology classes in $\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$ (in our case now, $k = 7$) are generated by a generator h in $H^2(\mathbb{CP}^2)$ and cohomology classes e_i representing exceptional spheres for $1 \leq i \leq k$, as we know that

$$\begin{aligned} h \cdot e_i &= 0 && \text{for all } i \\ e_i \cdot e_i &= -1 && \text{for all } i \\ e_i \cdot e_j &= 0 && \text{for all } i \neq j \end{aligned}$$

With the Poincaré duality, we are able to pass from cohomology classes to homology classes and again denote the Poincaré duality of h and e_i by the same notations in Park's paper.

Then we come to the difficulties and solutions in the paper:

1. How to build a manifold which is symplectic and homeomorphic to $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$? By lemma 3.1 in [P1], we are able to show that $E(1) = \mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$ contains an $\widetilde{E}_6$-singular fiber where it can be regarded as smooth 4-manifold obtained by plumbing disk bundles according the tree below (refer to section(3.4) for plumbing).



Furthermore, the second (co)homology classes $[S_i]$ $(1 \leq i \leq 7)$ of the 2-spheres S_i embedded in $\widetilde{E_6}$-singular fiber can be represented by

$$\begin{aligned}
[S_1] &= e_4 - e_7 \\
[S_2] &= e_1 - e_4 \\
[S_3] &= h - e_1 - e_2 - e_3 \\
[S_4] &= e_2 - e_5 \\
[S_5] &= e_5 - e_9 \\
[S_6] &= e_3 - e_6 \\
[S_7] &= e_6 - e_8
\end{aligned}$$

where h denotes a generator of $H_2(\mathbb{CP}^2;(Z))$ $(H^2(\mathbb{CP}^2;(Z))$ respectively) and each e_i denotes the (co)homology class represented by the i^{th} exceptional sphere in $E(1)$.

Then, as in proposition (3.1) in [P1], blowing-up 4 times at 4 double points in fishtail fibers gives embedded spheres whose homology classes are represented by $f-2e_{10}$, $f-2e_{11}$, $f-2e_{12}$ and $f-2e_{13}$, where $f = 3h - (e_1 + \cdots + e_9)$ is the homology class of the elliptic fiber in $E(1)$ and e_{10} to e_{13} are exceptional spheres obtained by those 4 blow-up operations. By resolving intesection points between $f-2e_{10}$, $f-2e_{11}$, $f-2e_{12}$, $f-2e_{13}$ and e_9, then we get an embedded sphere S

which has homology class

$$(f - 2e_{10}) + (f - 2e_{11}) + (f - 2e_{12}) + (f - 2e_{13}) + e_9 = 4f + e_9 - 2(e_{10} + \cdots + e_{13})$$

Considering S_1, S_2, S_3, S_4, S_5 and S these six spheres embedded in $E(1)\#4\overline{\mathbb{CP}}^2$, since all operations above can be performed symplectically, those spheres are embedded symplectically. Then we quote a theorem from Symington [Sy1]:

Theorem 4.2.1. *Suppose X is a symplectic 4-manifold containing a configuration C_p with a symplectic 2-form ω. If all 2-spheres u_i in C_p are symplectically embedded and intersect positively, then the rational blowdown manifold $X_p = (X - int\ C_p) \bigcup_{L(p^2,1-p)} B_p$ admits a symplectic 2-form ω_p which is symplectomorphic to ω when we consider their restriction on $X - int\ C_p$.*

By this result, we see that C_7 is symplectially embedded in $E(1)\#4\overline{\mathbb{CP}}^2$ and the resulting manifold after performing rational blowdown is still a symplectic manifold which is our target, denoted by X_7.

A remark is that constructing C_7 requires the intersection between those spheres must follow the tree structure as mentioned before, i.e.

- intersection of adjacent spheres $= +1$, otherwise it is 0.
- self-intersection of those spheres are indicated in the tree.

Obviously, we are almost success by taking those five spheres from the $\widetilde{E_6}$-singular fiber. It also provides hints for us to find the last sphere by the constrains from the intersections of the spheres.

2. How to prove X_7 is homeomorphic to $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$?

Before working on this question, we first recall two results. Let S be a simply connected 4-manifold with an almost complex structure, then:

- $c_2[S] = \chi(S) = 2 + rk(S)$

  where $\chi(S)$ is the Euler characteristic of S. The first equality is just the property of chern number and the second equality is the definition of $\chi(S)$. Since S is simply connected, $\dim(H^0(S))$=$\dim(H^4(S))$=1 and $\dim(H^1(S))$=$\dim(H^3(S))$=1. Thus we have:

$$\begin{aligned}\chi(S) &= \sum_{i=0}^{4}(-1)^i\dim(H^i(S)) \\ &= 1 + 0 + \dim(H^2(S)) + 0 + 1 \\ &= 2 + rk(S)\end{aligned}$$

- $c_1^2[S] = 3\sigma(S) + 2\chi(S)$

  This is a result from W.-T. Wu which is explained in [GS] (Theorem 1.4.15).

Therefore, determining the chern numbers $c_2[S]$ and $c_1^2[S]$ is just the same as determining $rk(S)$ and $\sigma(S)$, i.e.

$$rk(S) = c_2[S] - 2 \qquad \text{and} \qquad \sigma(S) = \frac{1}{3}(c_1^2[S] - 2c_2[S])$$

Combining these results and theorem (2.2.1), (2.2.2), we have the following corollary:

Corollary 4.2.2. *Let S_1 and S_2 be simply connected 4-manifolds with almost complex structures, if both intersection forms of them are indefinite and have the same parity, also $c_2[S_1] = c_2[S_2]$, $c_1^2[S_1] = c_1^2[S_2]$, then S_1 and S_2 are homeomorphic.*

With this corollary, we start our calculations:

(a) $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$

We know that $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$ is simply connected. The intersection form is $\langle 1\rangle\oplus 7\langle -1\rangle$, so $b_2^+(\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2) = 1$ and $b_2^-(\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2) = 7$. The intersection form is indefinite as there are both positive and negative eigenvalues. By simple calculations, $c_2 = 10$ and $c_1^2 = 2$. Obviously,

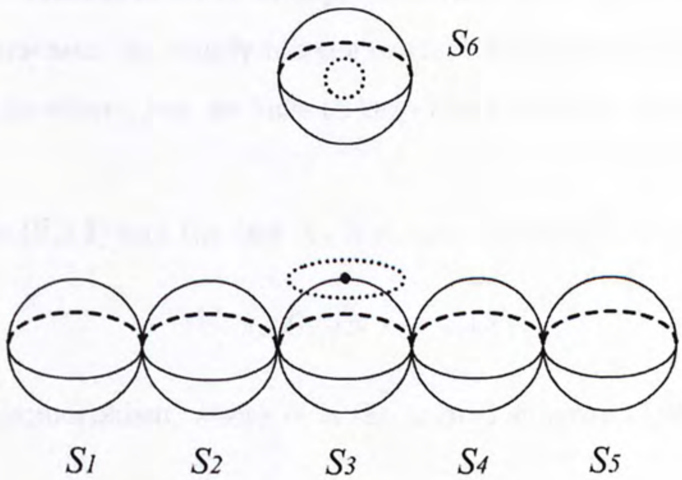
$$\begin{aligned} h \cdot h &= 1 \\ &\equiv 1 \pmod 2 \end{aligned}$$

It implies the intersection form is odd (h is a generator of $H_2(\mathbb{CP}^2)$).

(b) X_7

Note that X_7 is symplectic, so it carries a canonical almost complex structure. Recall that $X_7 = (E(1)\#4\overline{\mathbb{CP}}^2 - \text{int } C_7) \bigcup_{L(49,-6)} B_7$, for simplicity, we denote $(E(1)\#4\overline{\mathbb{CP}}^2 - \text{int } C_7)$ by X_0 and $E(1)\#4\overline{\mathbb{CP}}^2$ by X. Since X is simply connected, $X_0 = (X - \text{int } C_p)$ has only the lens space $L(49, -6)$ as boundary (see rational blowdown in section(3.5)), the only possible contribution to the fundamental group $\pi_1(X_0)$ is a generator of the fundamental group of the lens space.

We come back to the configuration C_7, see the figure below, the attached disk (dotted cicle) represents a fiber of the disk bundle over S_3 and it's boundary circle is exactly a generator of $\pi_1(L(49, -6))$. However, that circle is glued to the dotted circle on S_6 which is then contractible in X_0. Therefore, X_0 is simply connected.



Then we try to calculate the fundamental group of X_7 by using Van Kampen's Theorem. Now we have $X_7 = X_0 \cup B_7$, but X_0 and B_7 are not open, therefore a little modifications are needed to be made.

Firstly, we have to show the tubular neighborhood of the boundary $L(49, -6)$ is actually a product as $L(49, -6) \times [-1, 1]$ ($\partial B_7 = L(49, -6) \times \{0\} = \partial X_0$). Then we extend both X_0 and B_7 a little bit, i.e.

$$X_0 \text{ is extended to } \cup L(49, -6) \times [0, 1)$$

$$B_7 \text{ is extended to } \cup L(49, -6) \times (-1, 0]$$

where $L(49, -6) \times [0, 1) \subseteq B_7$ and $L(49, -6) \times (-1, 0] \subseteq X_0$. However, the two resulting manifold are open and they are homeomorphic to X_0 and B_7 separately (so having same fundamental group). Also their intersection is $L(49, -6) \times (-1, 1)$ which is homeomorphic to $L(49, -6)$.

With the above settings, Van Kampen's Theorem can be now applied. Note that the intersection of those two pieces is $L(49, -6) \times (-1, 1)$ whose fundamental group equals to $L(49, -6)$. To avoid those complicated notations of this thicken process in order to apply Van Kampen's Theorem and Mayer-Vietoris sequences, we simply use the original decomposition $X_7 = X_0 \cup B_7$ and so as the others, but we have to keep this process in mind.

By corollary(2.3.2) and the fact X_0 is simply connected, we have

$$k : \pi_1(B_7)/N \longrightarrow \pi_1(X_7)$$

is a group isomorphism, where N is the normal subgroup generated by the image of

$$i : \pi_1(L(49,-6)) \longrightarrow \pi_1(B_7)$$

where i is induced from the inclusion of $L(49,-6)$ in B_7. However, $\pi_1(B_7) = \mathbb{Z}_7$ and so N is exactly isomorphic to $\mathbb{Z}_7$. Therefore, $\pi_1(X_7) = 1$ and we conclude that X_7 is simply connected.

Consider the following Mayer-Vietoris sequence:

$$\begin{aligned} \cdots &\to H^1(L(49,-6);\mathbb{Q}) \to H^2(X_7;\mathbb{Q}) \to H^2(X_0;\mathbb{Q}) \oplus H^2(B_7;\mathbb{Q}) \\ &\to H^2(L(49,-6);\mathbb{Q}) \to \cdots \end{aligned}$$

As $H^1(L(49,-6);\mathbb{Q}) = H^2(L(49,-6);\mathbb{Q}) = 0$ and $H^2(B_7;\mathbb{Q}) = 0$, so $H^2(X_7;\mathbb{Q})$ is isomorphic to $H^2(X_0;\mathbb{Q})$. It immediately implies that

$$rk(X_7) = rk(X_0)$$

Similarly, we have

$$\begin{aligned} \cdots &\to H^1(L(49,-6);\mathbb{Q}) \to H^2(X;\mathbb{Q}) \to H^2(X_0;\mathbb{Q}) \oplus H^2(C_7;\mathbb{Q}) \\ &\to H^2(L(49,-6);\mathbb{Q}) \to \cdots \end{aligned}$$

and as a result, $H^2(X;\mathbb{Q})$ is isomorphic to $H^2(X_0;\mathbb{Q}) \oplus H^2(C_7;\mathbb{Q})$. It immediately implies that

$$\begin{aligned} rk(X) &= rk(X_0) + rk(C_7) \\ 14 &= rk(X_0) + 6 \\ rk(X_0) &= 8 \end{aligned}$$

Therefore, we get

$$\begin{aligned} c_2[X_7] &= 2 + rk(X_7) \\ &= 2 + 8 \quad (\because rk(X_7) = rk(X_0) = 8) \\ &= 10 \end{aligned}$$

Then we move to calculation of $c_1^2[X_7]$. Following the idea in [P1], for C_7, let

$$\{\gamma_i \in H^2(C_7; \mathbb{Q}) : 1 \leq i \leq 6\}$$

be the dual basis of $\{u_i \in H_2(C_7; \mathbb{Z}) : 1 \leq i \leq 6\}$, that means $\langle \gamma_i, u_j \rangle = \delta_{ij}$.

Then the intersection form

$$
\begin{aligned}
Q_{C_7} &= (\gamma_i, \gamma_j) \\
&= (u_i, u_j)^{-1} \\
&= \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -9 \end{pmatrix}^{-1} \\
&= -\frac{1}{49}\begin{pmatrix} 41 & 33 & 25 & 17 & 9 & 1 \\ 33 & 66 & 50 & 34 & 18 & 2 \\ 25 & 50 & 75 & 51 & 27 & 3 \\ 17 & 34 & 51 & 68 & 36 & 4 \\ 9 & 18 & 27 & 36 & 45 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}
\end{aligned}
$$

By the way, suppose that K and K_7 is the canonical classes of X and X_7 respectively. By the argument about splitting of cohomology classes, we get

$$
\begin{aligned}
c_1(X) &= K = K|_{X_0} + K|_{C_7} \\
c_1(X_7) &= K_7 = K|_{X_0} + K|_{B_7} = K|_{X_0} \\
\therefore \quad K &= K_7 - K|_{C_7}
\end{aligned}
$$

Now, we know that $[\gamma_1, \cdots, \gamma_6]$ is a basis for $H^2(C_7, \mathbb{Q})$. Let $K|_{C_7} = a_1\gamma_1 + \cdots + a_6\gamma_6$, then

$$
\begin{aligned}
K|_{C_7} \cdot u_j &= \sum_{i=1}^{6} a_i \langle \gamma_i, u_j \rangle \\
&= a_j
\end{aligned}
$$

Recall that $K = -3h + (e_1 + \cdots + e_{13})$ and note that $K|_{C_7} \cdot u_j = K \cdot u_j$, so we have

$$\begin{aligned} K|_{C_7} &= (K \cdot u_1)\gamma_1 + (K \cdot u_2)\gamma_2 + \cdots + (K \cdot u_6)\gamma_6 \\ &= 7\gamma_6 \end{aligned}$$

Finally, we have

$$\begin{aligned} -4 &= c_1^2[X] \\ &= K \cdot K \\ &= K_7 \cdot K_7 + K|_{C_7} \cdot K|_{C_7} \qquad (\because K_7 \cdot K|_{C_7} = 0) \\ &= c_1^2[X_7] + 49\gamma_6 \cdot \gamma_6 \\ &= c_1^2[X_7] - 6 \\ \therefore \quad c_1^2[X_7] &= 2 \end{aligned}$$

Try to calculate back the signature and rank, we find that

$$rk(X_7) = 8 \neq \pm\sigma(X_7) = \pm 6$$

Therefore, the intersection form of X_7 is again indefinite. Consider the element $e_8 - 2e_{10}$, we can easily check that it is orthogonal to every element in the basis $\{\gamma_i \in H^2(C_7; \mathbb{Q}) : 1 \leq i \leq 6\}$, and thus it is in $H^2((X_7); \mathbb{Q})$. By the way,

$$(e_8 - 2e_{10}) \cdot (e_8 - 2e_{10}) = -5$$

which is odd, so the the intersection form of X_7 is odd.

Then we finish proving X_7 is homeomorphic to $\mathbb{CP}^2 \# 7\overline{\mathbb{CP}}^2$ by using the above calculation and corollary(4.2.2).

3. b_2^+ of the new manifold we constructed equals to 1. How to show that the Seiberg-Witten invariant is well-defined?

By modifying results in [LL2] and using results in [LL1], refer to lemma (3.2) and (3.3) in [P1]. The author is able to show that there is a unique symplectic structure on $E(1)\#4\overline{\mathbb{CP}}^2$ up to diffeomorphism and deformation. Also, the symplectic 2-form on $E(1)\#4\overline{\mathbb{CP}}^2$ can be represented by

$$ah - (b_1e_1 + \cdots b_{13}e_{13})$$

which satisfies $a \geq b_1 \geq \cdots \geq b_{13} \geq 0$ and $3a > b_1 + \cdots + b_{13}$.

Therefore, we are able to calculate the symplectic form

$$\begin{aligned}
\omega|_{C_7} &= ([\omega] \cdot u_1)\gamma_1 + \cdots + ([\omega] \cdot u_6)\gamma_6 \\
&= (b_4 - b_7)\gamma_1 + (b_1 - b_4)\gamma_2 + (a - b_1 - b_2 - b_3)\gamma_3 + (b_2 - b_5)\gamma_4 \\
&\quad +(b_5 - b_9)\gamma_5 + \{12a - 4(b_1 + \cdots + b_9) - 2(b_{10} + \cdots + b_{13}) + b_9\}\gamma_6
\end{aligned}$$

As a result, we get

$$\begin{aligned}
&K|_{C_7} \cdot [\omega|_{C_7}] \\
= \;& 7\gamma_6 \cdot \{(b_4 - b_7)\gamma_1 + (b_1 - b_4)\gamma_2 + (a - b_1 - b_2 - b_3)\gamma_3 + (b_2 - b_5)\gamma_4 \\
& +(b_5 - b_9)\gamma_5 + \{12a - 4(b_1 + \cdots + b_9) - 2(b_{10} + \cdots + b_{13}) + b_9\} \\
= \;& -\frac{1}{7}\{(b_4 - b_7) + 2(b_1 - b_4) + 3(a - b_1 - b_2 - b_3) + 4(b_2 - b_5) \\
& +5(b_5 - b_9) + 6(12a - 4(b_1 + \cdots + b_9) - 2(b_{10} + \cdots + b_{13}) + b_9)\} \\
= \;& -\frac{1}{7}\{75a - 25b_1 - 23b_2 - 27b_3 - 25b_4 - 23b_5 - 24b_6 - 25b_7 - 24b_8 \\
& -23b_9 - 12(b_{10} + \cdots + b_{13})\}
\end{aligned}$$

With this special properties and use the Mayer-Vietoris sequences like before, we have the followings (see lemma (2.1) and theorem (2.4) and (3.1) in [P1]):

$$K = K_7 - K|_{C_7} \qquad \text{and} \qquad [\omega] = [\omega_7] - [\omega|_{C_7}]$$

and so

$$\begin{aligned}
K_7 \cdot [\omega_7] &= K_7|_{X_0} \cdot [\omega_7|_{X_0}] \\
&= \{-3h + (e_1 + \cdots + e_{13})\} \cdot + \\
&\quad \frac{1}{7}\{75a - 25b_1 - 23b_2 - 27b_3 - 25b_4 - 23b_5 - 24b_6 - 25b_7 - 24b_8 \\
&\quad -23b_9 - 12(b_{10} + \cdot + b_{13})\} \\
&= \frac{1}{7}\{54a - 18b_1 - 16b_2 - 20b_3 - 18b_4 - 16b_5 - 17b_6 - 18b_7 - 17b_8 \\
&\quad -16b_9 - 5(b_{10} + \cdots + b_{13})\} \\
&> \frac{1}{7}\{2b_2 - 2b_3 + 2b_5 + b_6 + b_8 + 2b_9 + 13(b_{10} + \cdots + b_{13})\} \\
&\quad (\because\ 3a > b_1 + \cdots + b_{13}) \\
&\geq 0 \quad (\because\quad b_2 \geq b_3 \text{ and } b_i \text{ are all positive})
\end{aligned}$$

Thus we proved that $-K_7 \cdot [\omega_7] < 0$, hence refer to (1.2.4) the Seiberg-Witten invariant is well defined and by the result obtained about symplectic manifolds, the Seiberg-Witten invariant for the canoncial class K_7 equals to ± 1.

By all of the above, we conclude that X_7 is homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$.

Besides the above results, we also have a few remarks:

1. When we start from $E(1) = \mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$, we take advantage that $E(1)$ contains a $\widetilde{E_6}$-singular fiber so that it gives some hints to find embedding spheres with certain intersecting numbers for performing rational blowdown. However, suppose we start with S_1, S_2, S_3, S_6 and S_7 and construct one more embedding sphere S like what we did before, we can still get a symplectic manifold homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$. The author of [P1] could not determine whether those two are diffeomorphic or not.

2. How many times of blowup should be performed on $E(1) = \mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$? How to determine the integer p for the rational blowdown operation (replacing C_p by

B_p)? Fortunately, theorem(3.5.4) tells us that $C_p \cup \overline{B_p} = \#(p-1)\overline{\mathbb{CP}^2}$ and it does provide a hint. If we think it topologically, a manifold $X = X_0\#(p-1)\overline{\mathbb{CP}^2}$, we replace C_p in $\#(p-1)\overline{\mathbb{CP}^2}$ by B_p, then resulting manifold is $X_0\#B_p\#\overline{B_p}$ which is just X_0. In our case, X_0 should be $\mathbb{CP}^2\#7\overline{\mathbb{CP}^2}$ and X is the manifold obtained by blowing up $E(1)$ several times. Let k be the number of blowup performing on $E(1)$. Then we obtain this relation between k and p:

$$\begin{aligned} 9+k &= 7+(p-1) \\ k &= p-3 \end{aligned}$$

Furthermore, we have 5 embedded spheres with nice intersection numbers, the ideal situation should be finding one more spheres to form the configuration C_7 (the author of [P1] succeeded at the end). This time, we have $p = 7$ and $k = 4$.

4.3 Progress of Researches

$\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$

One ultimate goal of mathematicians is trying to find the minimal k such that $\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$ has exotic structures. We summarize the results as a table:

k	mathematicans	paper
$k \geq 9$	Friedman and Morgan	On the differeomorphism types of certain algebraic surfaces [FM]
9	Okonek and Van de Ven	Stable bundles and differentiable structures on certian algebra surfaces [OV]
8	D.Kotschick	On manifolds homeomorphic to $\mathbb{CP}^2\#8\overline{\mathbb{CP}}^2$ [K]
7	J.Park	Simply Connected Symplectic 4-Manifolds with $b_2^+ = 1$ and $c_1^2 = 2$ [P1]
6	A.Stipsicz and Z.Szabó	An exotic smooth structure on $\mathbb{CP}^2\#6\overline{\mathbb{CP}}^2$ [SS1]
5	J.Park, A.Stipsicz and Z.Szabó	Exotic smooth structures on $\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$ [PSS]
6,7,8	R.Fintushel and R.Stern	Double node neighborhoods and families of simply connected 4-manifolds with $b^+ = 1$ [FS1]
3	A. Akhmedov, S. Baldridge and B.Doug Park	Exotic smooth structures on small 4-manifolds[ABP]

For the first three papers, results were not obtained by using Seiberg-Witten invariants. For $k = 7$, the techniques was showed in last subsection. Basically, for the cases of $k = 5, 6$ are just similar, but instead of rational blowdown, generalized rational blowdown was developed. Here we briefly explain what generalized rational blowdown is.

Generalized Rational Blowdown

We know that the lens spaces $L(p^2, 1-p)$ can bound a rational homology ball B_p by Kirby Calculus. In general, except this type of lens spaces, we have many more types of lens spaces can bound a rational homology ball, refer to [CH]. In particular, for the lens space $L(p^2, pq-1)$, where $p \geq q \geq 1$ and p, q are relatively prime, can also bound a rational homology ball. By plumbing disc bundles over 2-spheres in a 4-manifold along a linear graph like before such that continued fraction of the self-intersection numbers gives us $-\frac{p^2}{pq-1}$, and then replacing the whole structure by the rational homology ball, this operation is called generalized rational blowdown, for $q = 1$, it just reduces to the rational blowdown. Furthermore, we need to ensure this opeation can be performed symplectically, fortunately, we survived by the result from [Sy2].

With this generalized rational blowdown techniques, the authors tried to blowup $E(1)$ several times and that showed there were actually immersed spheres with intersecting number as required in the linear graph. After replacing the plumbing manifold by a rational homology ball, the resulting 4-manifold was what we wanted.

Combining technique of knot surgery in a double node neighborhood with a particular form of generalized rational blowdown, the paper [FS1] in the table showed for $k = 6, 7, 8$, $\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$ has infinitely many exotic structures. By using a technique called Luttinger surgery in a recent paper [ABP] enable the authors to find an exotic structure for $k = 3$. As we mentioned before, the minimum number of k such that $\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$ has exotic structure is still an open question.

$m\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$ **and** $m > 1$

Again recalling the result of Gromov and Lawson (Theorem A in [GL]) implies that every manifold of the type

$$m\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$$

admits a metric of positive curvature. It motivates us to go on searching exotic structures of manifolds in this form. However, Seiberg-Witten invariants are only defined when b_2^+ of the manifold is odd. This restricts our attention to those manifolds in the form of $m\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$ with m is odd. The following table summarizes results for $m \geq 3$:

m, k	mathematicans	paper
m odd ≥ 3, $k \geq m+7$	J. Park	Exotic smooth structures on 4-manifolds. II [P2]
$m = 3$, $k \geq 10$	B. Park	Constructing infinitely many smooth structures on $3\mathbb{CP}^2\#n\overline{\mathbb{CP}}^2$ [PA]
$m = 3$, $k = 9$	A. Stipsicz and Z. Szabó	Small exotic 4-manifolds with $b_2^+ = 3$ [SS2]
$m = 3$, $k = 8$	J. Park	Exotic structures on $3\mathbb{CP}^2\#8\overline{\mathbb{CP}}^2$ [P3]
$m = 3$, $k = 5$	A. Akhmedov, S. Baldridge and B.Doug Park	Exotic smooth structures on small 4-manifolds[ABP]

Remark

Like the example $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$, we finally found a symplectic manifold homeomorphic but not diffeomorphic to it. However, this method only tells us there exists an exotic structure but nothing about the amount, whether there are finitely many or infinitely many distinct smooth structures or even uncountably many distinct smooth structures. In order to obtain a family of homeomorphic 4-manifolds, new methods other than our example have to be introduced and we will cover some of them next.

Chapter 5

Gluing Results in Seiberg-Witten Theory

Methods to produce an infinite set of homeomorphic 4-manifolds which have different smooth structures developed by researchers basically followed this strategy: We first define a new version of Seiberg-Witten invariant which combines all the Seiberg-Witten invariants into one, then develop an operation or surgery and perform it to obtain a infinite set of homeomorphic 4-manifolds, calculating the new invariants of them and show they are all distinct, so we obtain a 4-manifold with exotic structures.

In the coming sections, we will review the Seiberg-Witten invariant and introduce some surgeries and give examples of manifolds which have infinitely many exotic structures.

5.1 Revisit of Seiberg-Witten Invariants

Suppose M is a smooth closed oriented 4-manifold with $b_2^+(M) > 1$. If $H_1(M;\mathbb{Z})$ has no 2-torsion, then there is a natural identification of the $spin^c$ structures of M with the characteristic elements of $H^2(M;\mathbb{Z})$, refer to chapter 10 of [S]. Thus we are able to regard the Seiberg-Witten invariants as an integer-valued function:

$$SW_M : \{k \in H^2(M;\mathbb{Z}) | k \equiv w_2(TM) \pmod 2\} \longrightarrow \mathbb{Z}$$

In this whole chapter, all manifolds we are going to discuss has no 2-torsion for the first homology group, e.g. simply connected manifold. Recall that a *spin*^{c} structure β with $SW_M(\beta) \neq 0$ is called a basic class. As a matter of fact, there are only finitely many basic classes. Furthermore, if β is a basic class, then $-\beta$ is also a basic class and they are related by

$$SW_M(-\beta) = (-1)^{\frac{(e+\sigma)(M)}{4}} SW_M(\beta)$$

where $e(M)$ is the Euler characteristic of M and $\sigma(M)$ is the signature of M. Actually the sign difference between $SW_M(-\beta)$ and $SW_M(\beta)$ comes from the orientation and related by the factor $(-1)^{\frac{(e+\sigma)(M)}{4}}$.

With the above, we let $\{\pm\beta_1, \cdots, \pm\beta_n\}$ be the set of nonzero basic classes for M and define the Seiberg-Witten Series of M to be the Laurent polynomial

$$\begin{aligned} \mathscr{SW}_M &= \sum SW_M(\beta) exp(\beta) \quad \text{summation over all basic classes } \beta \\ &= b_0 + \sum_{j=1}^{n} b_j (exp(\beta_j) + (-1)^{\frac{(e+\sigma)(M)}{4}} exp(-\beta_j)) \end{aligned}$$

where $b_0 = SW_M(0)$ and $b_j = SW_M(\beta_j)$. Clearly, two diffeomorphic manifolds have same Seiberg-Witten Series and so it can be considered as an invariant. One of the advantages of writing Seiberg-Witten invariant in this form is that we can consider all basic classes at once. By letting $t_\beta = exp(\beta)$, we have the most important property $t_{\alpha+\beta} = t_\alpha t_\beta$ and so we have

$$\mathscr{SW}_M = b_0 + \sum_{j=1}^{n} b_j (t_{\beta_j} + (-1)^{\frac{(e+\sigma)(M)}{4}} t_{\beta_j}^{-1})$$

By corollary(1.3.8), K3 surface has only one basic class which is the anti-canonical class. As a result

$$\mathscr{SW}_{K3} = 1$$

with suitable choice of orientation.

5.2 Fiber Sums and its Generalization

Fiber Sums is explained in deep in [S] and [GS], we quickly review them.

Recall that $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$, it has a algebraic description. Firstly, pick a $\mathbb{CP}^2$ and choose two generic homogeneous polynomial of degree 3 on it, so they are two curves on $\mathbb{CP}^2$ and intersect each other at 9 points transversely. We blowup at those 9 points and get $E(1)$ also we have a natural projection of $E(1)$ on $\mathbb{CP}^1$ such that the preimage of points on $\mathbb{CP}^1$ are torus except finite number of them.

Now we may take two $E(1)$ and a generic fiber (i.e. the fiber is a torus) from each of them, then take out a small neighborhood of each torus so they have a boundary $T^2 \times S^1$. We glue their boundaries together (glue the torus part by identity map and S^1 part by the map $z \mapsto \overline{z}$) and get a new manifold, called $E(2)$. This gluing operation is called fiber sum. Of course, this operation can be extended to n copies of $E(1)$ and get $E(n)$.

The key point of such a gluing is that we can take out a solid $T^2 \times D^2$ from each 4-manifold and glue them together along the same boundary. Therefore, for $1 \leq i \leq 2$, if T_i is a torus embedded in 4-manifolds M_i with zero self-intersection, then their normal bundle in M_i is trivial, so we are able to take out a solid $T_i \times D^2$ from M_i and the boundary of the remaining part is $T_i \times S^1$. Fixing a orientation preserving diffeomorphism ϕ between the two tori T_1 and T_2, together with the complex conjugate map $\psi : z \mapsto \overline{z}$, we have a orientation reversing diffeomorphism $(\phi, \psi) : T_1 \times S^1 \longrightarrow T_2 \times S^1$. Finally, gluing the remaining part of M_1 and M_2 by using this diffeomorphism to get a new manifold, this gluing is called generalized fiber sum and denoted by

$$M_1 \#_{T_1 =_\phi T_2} M_2$$

In order to write a nice formula for the Seiberg-Witten series, the requirement that the two tori have zero self-intersection is not enough. We have to introduce what near-cusp embedded torus (or called c-embedded torus) is.

Suppose M is a simply connected smooth 4-manifold with $b_2^+ > 1$. We define a cusp in M to be a PL embedded 2-sphere with zero self-intersection and a single nonlocally flat point whose neighborhood is the cone on the right hand trefoil knot. If M is an elliptic surface, the definition of a cusp exactly coincides with cusp fiber in elliptic surface. The neighborhood N of a cusp in M is called a cusp neighborhood (which can be imagined as the manifold obtained by performing 0-framed surgery on a trefoil knot in the boundary of the 4-ball).

Definition 5.2.1. Suppose T is a smoothly embedded torus in M representing a nontrivial homology class $[T]$ (homologically nontrivial). If T is a smooth fiber in a cusp neighborhood N, then T is said to be a near-cusp embedded torus (c-embedded torus).

Theorem 5.2.2. *Let M_1, M_2 be two 4-manifolds with $b_2^+ \geq 2$ and let T_1, T_2 be two homologically nontrivial near-cusp embedded tori in M_1 and M_2 respectively. Then,*

$$\mathscr{SW}_{(M_1 \#_{T_1=T_2} M_2)} = \mathscr{SW}_{M_1} \cdot \mathscr{SW}_{M_2} \cdot (e^{T_1} - e^{-T_1}) \cdot (e^{T_2} - e^{-T_2})$$

where T_1 and T_2 in the equation represent the cohomology classes of the images of the two tori in $M_1 \#_{T_1=T_2} M_2$.

Example 5.2.3. It is known that E(2) is actually a K3 surface (refer to section 3.3 of [S] or any references about elliptic surfaces) and so $\mathscr{SW}_{E(2)} = 1$, also it has $b_2^+ \geq 2$. Therefore we can obtain $\mathscr{SW}_{E(4)}$ by

$$\begin{aligned} \mathscr{SW}_{E(4)} &= \mathscr{SW}_{E(2) \#_{\text{torus fiber } T} E(2)} \\ &= \mathscr{SW}_{E(2)} \cdot \mathscr{SW}_{E(2)} \cdot (e^T - e^{-T})^2 \\ &= e^{2T} - 2e^0 + 2e^{-2T} \quad (\text{since } \mathscr{SW}_{E(2)} = 1) \end{aligned}$$

Thus, $E(4)$ has basic classes $\pm 2F$ and 0, with values 1 and -2 respectively.

Unfortunately, $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$ has $b_2^+ = 1$, therefore formula for gluing $E(1)$ has to be treated separately:

Theorem 5.2.4. *Let M be a 4-manifold with $b_2^+ \geq 2$ and let T is homologically non-trivial near-cusp embedded torus in M. Then,*

$$\mathscr{SW}_{(M \#_{T=F} E(1))} = \mathscr{SW}_M \cdot (e^T - e^{-T})$$

Example 5.2.5. Again recall that $E(n) = E(2)\#_{\text{torus fiber } T}(n-2)E(1)$ for $n \geq 2$. Then by the above theorem, we have

$$\mathscr{SW}_{E(n)} = (e^T - e^{-T})^{n-2}$$

By putting $n = 4$, we get $\mathscr{SW}_{E(4)} = (e^T - e^{-T})^2$ which coincides with the result before.

5.3 Logarithmic transformations and its Generalization

In 3-manifolds, as our discussion before, we have Dehn surgery to construct new manifolds. The spirit of it is taking out a solid torus, twisting it and gluing back. Logarithmic transformations can be regarded as an analogue in 4-manifolds. Details appear in standard references like [S] and [GS], here we only quickly revise it for later applications.

Given a closed 4-manifold M and a torus T embedded in M with zero self-intersection. Then a tubular neighborhood of T can be locally described as $T^2 \times D^2$ (since $T \cdot T = 0$) and so the boundary is $T^2 \times S^1 = S^1 \times S^1 \times S^1 = T^3$. We cut it out of M and glue it back by using an orientation reversing diffeomorphism from T^3 onto itself.

Let $\{\mu, \alpha_1, \alpha_2\}$ be a basis in $H_1(T^3; \mathbb{Z})$, with $T^3 = T^2 \times S^1$ and μ is the class of $\{*\} \times S^1$, while α_1 and α_2 correspond to the two classes coming from a fixed splitting of the torus. Since every self-diffeomorphism of T^3 is isotopic to a linear self-diffeomorphism, so suppose that ϕ is an orientation reversing diffeomorphism from T^3 onto itself, just like Dehn surgery, we have

$$\phi_*(\mu) = p\mu + m\alpha_1 + n\alpha_2$$

for some integers m, n and p such that they have no common divisor except 1. Furthermore, it can proved that all such diffeomorphism are completely determined by the image of μ. Therefore, in conclusion, given a triple of integers (p, m, n) with no common divisor except 1, there is a unique 4-manifold

$$M_{(p,m,n)} = (M - \text{int } (T \times D^2)) \cup_\phi (T^2 \times D^2)$$

and such operation is called a generalized logarithmic transformation. To simplify notation, we will use M_0 to denote the remaining part of M after taking out the neighborhood of T, i.e. $M_0 = M - \text{int } (T \times D^2)$.

Originally, logarithmic transformation is defined on elliptic surfaces and performed on a regular torus fiber, see [S]. Generalized logarithmic transformation is just a generalization of it and they coincide when M is an elliptic surface.

Theorem 5.3.1. *Let E be an elliptic surface containing a cusp fiber and T be a regular fiber of E. Then, for every m, n and p with no common divisor except 1, we have*

$$E_p \cong E_{(p,m,n)}$$

where E_p denotes the logarithmic transformation of multiplicity p.

Now, we come to the formula of Seiberg-Witten Series for performing generalized logarithmic transformation, proved in [MMS].

Theorem 5.3.2. *Suppose M is a smooth compact oriented 4-manifold which has $b_2^+ \geq 2$ and T is a torus embedded in M with zero intersection. If κ_0 is a spin^c structure on $M_{(p,m,n)}$ that restricts nontrivially to the plugged in $T^2 \times D^2$, then $SW_{M_{(p,m,n)}}(\kappa_0) =$*

0. *Furthermore, for each spin^c structure κ_0 that restricts trivially to the boundary $\partial M_0 = T \times S^1$, let $V_{(p,m,n)}(\kappa_0)$ be the set of isomorphism classes of spin^c structures over $M_{(p,m,n)}$ whose restriction to M_0 is equal to κ_0. Then we have*

$$\begin{aligned}\sum_{\kappa\in V_{(p,m,n)}(\kappa_0)} SW_{M_{(p,m,n)}}(\kappa) &= p\cdot \sum_{\kappa\in V_{(1,0,0)}(\kappa_0)} SW_{M_{(1,0,0)}}(\kappa)\\ &+ m\cdot \sum_{\kappa\in V_{(0,1,0)}(\kappa_0)} SW_{M_{(0,1,0)}}(\kappa) + n\cdot \sum_{\kappa\in V_{(0,0,1)}(\kappa_0)} SW_{M_{(0,0,1)}}(\kappa)\end{aligned}$$

By computing Seiberg-Witten invariants in a champer, this theorem can be extended to $b_2^+ = 1$.

The proof of this theorem involves defining relative Seiberg-Witten invariants on 4-manifold with boundary T^3 like the ordinary invariants. Then imposing a non-negative metric on the part $T^2 \times D^2$. By stretching M_0 near the boundary T^3 and make it into a cylindrical end (i.e. $T^3 \times [0,\infty)$) so that one can impose a metric on the manifold $M_{(p,m,n)}$ which is compatible with the metric on M_0 and the metric on $T^2 \times D^2$. With this nice metric, the solution space can be figured out and therefore the Seiberg-Witten invariants are related together as the above formula.

If the generalized logarithmic transformation is performed on a near-cusp embedded torus, the formulas for relating Seiberg-Witten Series is even nicer, which is proved by R. Fintushel and R. Stern.

Theorem 5.3.3. *Let M be a 4-manifold and T be a near-cusp embedded torus which is homologically nontrivial and torsion free, then we have:*

$$\begin{aligned}\mathcal{SW}_{M_{(0,m,n)}} &= 0 \quad \textit{and}\\ \mathcal{SW}_{M_{(p,m,n)}}\cdot(e^{T_p} - e^{-T_p}) &= \mathcal{SW}_M\cdot(e^T - e^{-T})\end{aligned}$$

where T_p is the class of the new plugged in torus in $M_{(p,m,n)}$. Furthermore, it can be proved that

$$T = p\cdot T_p$$

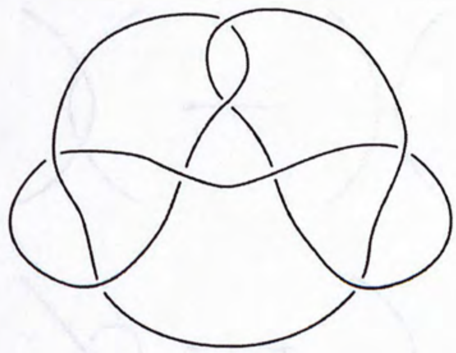
so the second equation can be rewritten as

$$\mathscr{SW}_{M_{(p,m,n)}} = \mathscr{SW}_M \cdot (e^{(p-1)T_p} + e^{(p-2)T_p} + \cdots + e^{-(p-1)T_p})$$

5.4 Knot Theory and Alexander Polynomials

In the last section, performing generalized logarithmic transformation involves gluing back a product of a torus and disk $T^2 \times D^2$ along T^3. However, it is possible for us to glue back other 4-manifold with boundary T^3. One option is gluing back a manifold with boundary $K \times S^1$ where K is a knot. In the later sections, we will see how R. Fintushel and R. Stern used this method to produce an infinite family of homeomorphic 4-manifold with different smooth structures, and see how the method related to the Alexander polynomial of a knot K. Before reaching there, we first revise some knowledge in knot theory and what Alexander polynomial is. The following materials can be found in any references about knot theory or [L].

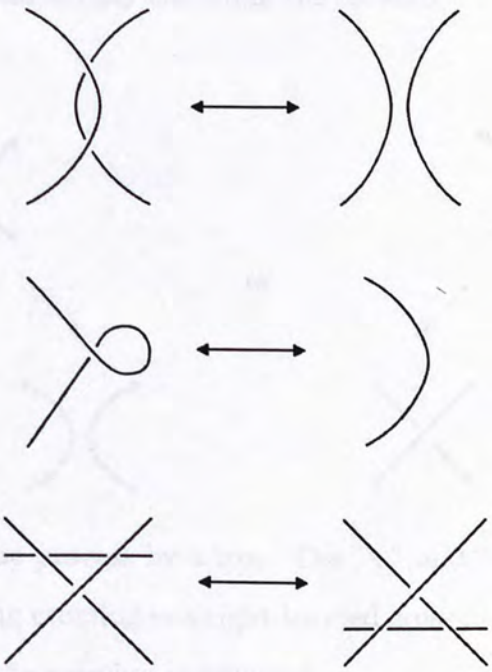
Intuitively a knot is a closed curve in $\mathbb{R}^3$ which does not intersect itself. Therefore a knot can be defined as the image of an injective continuous function $K : S^1 \longrightarrow \mathbb{R}^3$ (or to make the ambient space to be compact, we define as $K : S^1 \longrightarrow S^3$). In order to rule out wild knots, we can further impose the condition that the closed loop can be "approximated" by a polygonal closed curve. Also, by small perturbation if necessary, we can have a nice projection of the knot on a plane to obtain a knot diagram like the following one.



Preimage of each point of the plane contains at most two points, so it make sense to refer overcrossings and undercrossings. Also when we give an orientation of a knot, we can talk about right crossings and left crossings.

right-handed crossing *left-handed crossing*

Two knots K_1 and K_2 are said to be equivalent if there is an (orientation-preserving) homeomorphism from S^3 onto itself that takes K_1 to K_2. Different choices of projection of two equivalent knots in knot diagrams are related by a sequence of Reidemeister moves.

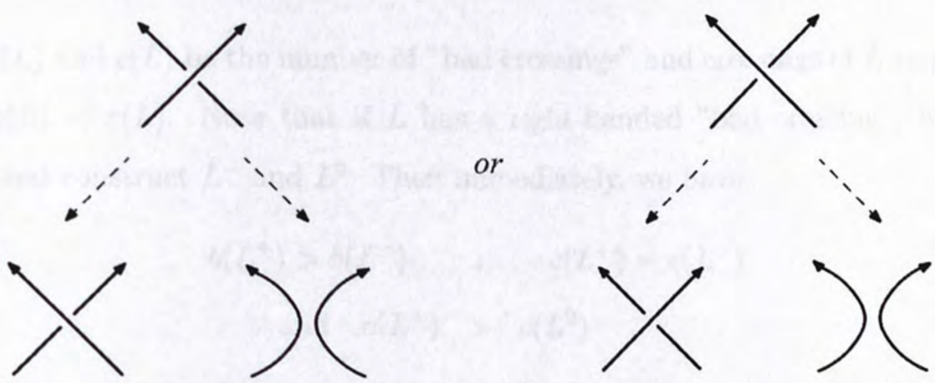


Besides knots, one of the most interesting object is links. A link is a collection of knots, we call them components of the link. In order to study how many times one component twists another, linking number of two components is defined. Given a link L and denote the components by $L_1, L_2, \cdots, L_n$, we give an orientation to each component. Then the linking number of L_i and L_j is defined by

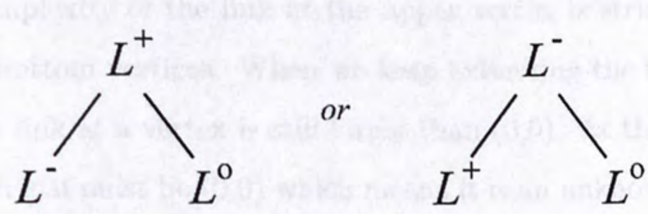
$$lk(L_i, L_j) = \frac{1}{2}(\text{number of right-handed crossings} - \text{number of left-handed crossings})$$

For the self-linking numbers $lk(L_k, L_k)$, it means the linking number of L_k and its small pertubation. Also it can proved that those linking numbers are independent from the choices of orientations of the components, hence they are well-defined. Furthermore, it is easy to show that $lk(L_i, L_j) = lk(L_j, L_i)$. As our expectation, if $lk(L_i, L_j) = 0$, we can separate two components in a diagram by performing Reidemeister moves.

Next we would like to define the (symmetrized) Alexander polynomial for a knot, but before that we need a lemma. Given a link L, if there is a right-handed (left-handed) crossing, then we can construct two links by switching the crossing to a left-handed (right-handed) crossing and simply canceling the crossing.



Also we can denote the process by a tree. The "+" and "-" signs of L^+ and L^- remind us the corresponding crossing is a right-handed crossing and left-handed crossing respectively, "0" means the crossing is removed.



Of course, we can keep doing that to those two new links and construct a tree.

Lemma 5.4.1. *Given a knot K, by performing the operation described above repeatly with suitable choices of crossing each time, one can obtain a tree such that each link at a leaf is an unknot or a split unlink (collection of unknots without crossing each other). The tree obtained is called a resolution tree.*

Proof. Given a link L, we can give an orientation and choose a base point for each knot component. Also we order the components and travel along those components according their ordering (start from their base points). Then when one travel along a knot component, one will meet those crossings twice, one overcrossing and one undercrossing. If we meet a crossing as undercrossing at the first time, then the crossing is called a "bad crossing". Furthermore, if there is no "bad crossing" in a link, then the link must be an unknot or a split unlink.

Let $b(L)$ and $c(L)$ be the number of "bad crossings" and crossings of L respectively, and so $b(L) < c(L)$. Note that if L has a right-handed "bad crossing", we denote $L = L^+$ and construct L^- and L^0. Then immediately, we have

$$b(L^+) > b(L^-) \quad ; \quad c(L^+) = c(L^-)$$
$$\text{and} \quad c(L^+) > c(L^0)$$

Now we consider $(c(L), b(L))$ as an ordered pair called complexity and arrange in the lexicographic ordering, then we have

$$(c(L^+), b(L^+)) > (c(L^0), b(L^0)) \quad \text{and} \quad (c(L^+), b(L^+)) > (c(L^-), b(L^-))$$

If the "bad crossing" is a left-handed crossing, we also have the similar argument. Therefore the complexity of the link at the upper vertex is strictly larger than those two links at the bottom vertices. When we keep extending the tree downwards if the complexity of the link at a vertex is still larger than (0,0). At the end, the complexity of the link at each leaf must be (0,0) which means it is an unknot or a split unlink. □

With the help of a resolution tree of a knot K, we are able to define the Alexander polynomial $\triangle_K(t)$. We begin from the leaves of the resolution tree. If it is an unknot U, we define $\triangle_U(t) = 1$ and if it is a split unlink S, we define $\triangle_S(t) = 0$. Then we use the skein relation

$$\triangle_{K^+}(t) = \triangle_{K^-}(t) + (t^{1/2} - t^{-1/2}) \cdot \triangle_{K^0}(t)$$

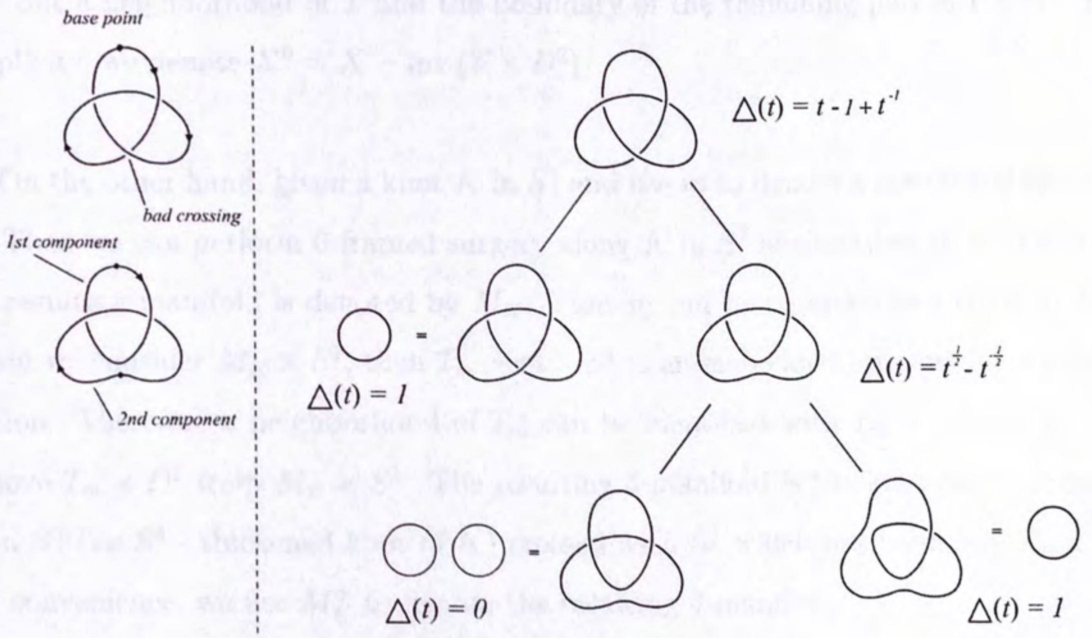
to calculate back the Alexander polynomial of K from the bottom of the resolution tree to the top.

Researches in knot theory tell us that Alexander polynomial of a knot is well defined, i.e. independent from the choice of projection and resolution tree, also an invariant under Reidemeister moves, hence it is a knot invariant. As a matter of fact, the Alexander polynomial we obtained is not really a polynomial but a Laurent polynomial. By induction, it can be proved that an Alexander polynomial of a knot must be in form of

$$a_0 + \sum_{j=1}^{n} a_j(t^j + t^{-j})$$

where a_0 and a_j are integers, so it is a symmetric Laurent polynomial. In knot theory, there are various ways to construct Alexander polynomials but the result differs from the one above by a multiple of t^m. Therefore, we may say the Alexander polynomial is unique after symmetrization (multiply the polynomial by t^m and make it in the form above). Conversely, given a Laurent polynomial of the above form, it must be the Alexander polynomial of some knot in S^3, hence there are infinitely many Alexander polynomial. We end up our discussion by an example.

Example 5.4.2. Now we try to construct a resolution tree and calculate the Alexander polynomial of the (left) trefoil knot.



Finally, we obtain the Alexander polynomial of trefoil knot $\triangle(t) = t - 1 + t^{-1}$.

5.5 Main Theorem

The materials of the following section basically come from the paper [FS3] by R. Fintushel and R. Stern. We first go through the idea of constructions. Given a manifold X with certain properties, a knot K, the authors tried to construct a new manifold X_K such that X_K is homeomorphic to X, but

$$\mathscr{SW}_{X_K} = \mathscr{SW}_X \cdot \triangle_K(t)$$

where $\triangle_K(t)$ is the Alexander polynomial of K and t is exponential of some cohomology class. Therefore, X_K is not diffeomorphic to X. Moreover, there are infinitely many Alexander polynomials implies we have an infinite family of homeomorphic manifolds with different differentiable structures. One of the manifolds satisfies those certain conditions is $K3$ surface.

Now, suppose X is a simply connected smooth 4-manifold with $b_2^+ > 1$ and T is a near-cusp embedded torus. Then T has zero intersection and again we are able to take out a neighborhood of T and the boundary of the remaining part is $T \times S^1$. For simplicity, we denote $X^0 = X - \text{int } (T \times D^2)$

On the other hand, given a knot K in S^3 and use m to denote a meridional circle to K. Then we can perform 0-framed surgery along K in S^3 as described in section(3.3), the resulting manifold is denoted by M_K. Thus m can be regarded as a circle in M_K. When we consider $M_K \times S^1$, then $T_m = m \times S^1$ is an embedded torus with zero intersection. Therefore a neighborhood of T_m can be identified with $T_m \times D^2$ and we can remove $T_m \times D^2$ from $M_K \times S^1$. The resulting 4-manifold is the knot complement of K in S^3 (i.e S^3- thickened knot of K) crossed with S^1 which has boundary $T_m \times S^1$. For convenience, we use M_K^0 to denote the resulting 4-manifold.

Since X^0 and M_K^0 have the same boundary, we glue them together to form

$$\begin{aligned} X_K &= X^0 \cup M_K^0 \\ &= X\#_{T=T_m}(M_K \times S^1) \quad \text{(notation from generalized fiber sum)} \end{aligned}$$

The two pieces are glued together so that the homology class $[* \times S^1]$ is preserved. To explain explicitly, for the thickened knot of K, there is a canonical choice of longitude so that it is contractible in the knot complement. Then a generator of $[* \times S^1]$ is mapped to that longitude.

Recall that homologically the knot complement is just a solid torus, when we take product of it and S^1, homologically it is just the product of a torus with D^2 and so M_K^0 has same homology with $T \times D^2$. Thus what the gluing operation does is constructing X_K so that it recovers X homologically, that means X and X_K have the same homology and intersection pairing. Furthermore, we know that $\pi_1(M_K \times S^1)$ is normally generated by the image of $\pi_1(T)$. Let us make an extra assumption that $\pi_1(X - T) = 1$. Then by corollary(2.3.2), X_K is simply connected.

X and X_K are both simply connected and having the same intersecting pairing, so by Freedman's classification theorem, X and X_K are homeomorphic. What the most interesting issue, which is the most difficult part to prove, is how their Seiberg-Witten Series related. In [FS3], the authors gave the following nice result:

Theorem 5.5.1. *Let X be a simply connected smooth 4-manifold with $b_2^+ > 1$ and let T be a smoothly near-cusp embedded torus in X such that $\pi_1(X - T) = 1$. By giving a knot K, a smooth 4-manifold X_K is constructed in the above way. Then X_K is homeomorphic to X and their Seiberg-Witten Series are related by*

$$\mathscr{SW}_{X_K} = \mathscr{SW}_X \cdot \triangle_K(t)$$

where $t = exp(2[T])$ and $\triangle_K(t)$ is the Alexander polynomial of the knot K.

Proof. The idea is constructing a resolution tree of the knot K as mentioned in section(5.4). When we keep track on $\mathscr{SW}_{K^+}$, $\mathscr{SW}_{K^-}$ and $\mathscr{SW}_{K^0}$, they are exactly related together like the skein relation. Also switching the crossing or canceling a crossing to form X_{K^-} (X_{K^+}) and X_{K^0} from X_{K^+} (X_{K^-}) can be achieved by performing a series of generalized logarithmic transformations on null-homologous tori, so the formulas from J. Morgan, T Mrowka and Z. Szabó help to prove the skein-like relations. □

In fact, from the result in [GM], there are three disjoint near-cusp embedded tori T_1, T_2 and T_3 in the K3 surface representing distinct homology classes $[T_j]$, $j = 1, 2, 3$. By theorem(5.5.1), given three knots K_j, $j = 1, 2, 3$, one can construct a new manifold $K3_{(K_1,K_2,K_3)}$ so that

$$\begin{aligned} \mathscr{SW}_{K3_{(K_1,K_2,K_3)}}(t_1, t_2, t_3) &= \mathscr{SW}_{K3} \cdot \triangle_{K_1}(t_1) \cdot \triangle_{K_2}(t_2) \cdot \triangle_{K_3}(t_3) \\ &= \triangle_{K_1}(t_1) \cdot \triangle_{K_2}(t_2) \cdot \triangle_{K_3}(t_3) \quad (\because \mathscr{SW}_{K3} = 1) \end{aligned}$$

where $t_j = exp(2[T_j])$ and $\triangle_{K_j}(t_j)$ is the Alexander polynomial of the knot K_j for $j = 1, 2, 3$.

Corollary 5.5.2. *The family of 4-manifolds homeomorphic to the K3 surface with distinct smooth structures is at least as rich as the family of Alexander polynomial, hence K3 surface has infinitely many distinct smooth structures.*

Furthermore, by using the results from C. Taubes, the authors were able to obtain results related to symplectic structures. Recall that an Alexander polynomial of a knot must be in form of

$$a_0 + \sum_{j=1}^{n} a_j(t^j + t^{-j})$$

If $a_n = \pm 1$, then it is said to be monic. If X is symplectic and T is a symplectically embedded torus, then X_K can be constructed to carry a symplectic structure if and only if $\triangle_K(t)$ is monic.

Before finishing our discussions, a remark have to be made to indicate that there is still a large dark area:

- Again this method by the authors only provided a way to find manifolds with infinitely many exotic structures, but it does not tell the exact amount. Even the worse, Alexander polynomial is not a complete knot invariant, that means there are some knots with the same Alexander polynomial. Given two knots K_1 and K_2 having the same Alexander polynomial, we still do not know whether those two 4-manifolds X_{K_1} and X_{K_2} are diffeomorphic or not. Moreover, there are too many restrictions on the manifold X which comes from defining Seiberg-Witten invariants (also Seiberg-Witten Series) and obtaining those gluing results, therefore it would be difficult for researchers to construct concrete examples.

Bibliography

[A1] S. Akbulut, On 2-dimensional homology classes of 4-manifolds, Math. Proc. Camb. Phil. Soc. 82 (1977), Birkhäuser

[ABP] A. Akhmedov, S. Baldridge and B.Doug Park, Exotic smooth structures on small 4-manifolds, arXiv:math.GT/0701664

[AK1] S. Akbulut and R. Kirby, Mazur manifolds, Michigan Math. J. 26 (1979), 259-284

[CH] A. Casson and J. Harer, Some homology lens spaces which bound rational homology balls, Pacific J. Math. 96 (1981) 23-36 MR0634760

[F] M. Freedman, The topology of four-dimensional manifolds, J. Diff. Geom. 17 (1982), 357-453

[FM] R. Friedman and J. Morgan, On the differeomorphism types of certain algebraic surfaces, I.J. Diff. Geom. 27, 297-369(1988)

[FS1] R. Fintushel and R. Stern, Double node neighborhoods and families of simply connected 4-manifolds with $b^+ = 1$, arXiv:math.GT/0412126

[FS2] R. Fintushel and R. Stern, Rational blowdowns of smooth 4-manifolds, J. Diff. Geom. 46 (1997), 181-235

[FS3] R. Fintushel and R. Stern, Knots, links, and 4-manifolds, Invent. math. 134, 363-400 (1998)

[FQ] M. Freedman and F. Quinn, Topology of 4-manifolds, Pinceton Mathematical Series 39, Princeton University Press, 1990

[GL] M. Gromov and H. B. Lawson, The classification of simply connected manifolds of positive scalar curvature, Annals of Math. 111 (1980), 423-434

[GM] R. Gompf and T. Mrowka, Irreducible 4-manifolds need not be complex, Ann. Math. 138, 61-111 (1993)

[GS] Robert E. Gompf and András I. Stipsicz, 4-Manifolds and Kirby Calculus, GSM Volume 20, American Mathematical Society

[J] J. Morgan, The Seiberg-Witten equations and applications to the topology of smooth four-manifolds, Math. Notes 44, Princeton University Press, 1996

[JJ] Jürgen Jost, Riemannian Geometry and Geometric Analysis, 3rd Edition, Springer

[K] D. Kotschicks, On manifolds homeomorphic to $\mathbb{CP}^2\#8\overline{\mathbb{CP}}^2$, Invert. Math. 95 (1989), 591-600

[L] Charles Livingston, Knot Theory, Washington, DC : Mathematical Association of America, 1993

[LL1] T. Li and A. Liu, Symplectic structures on ruled surfaces and a generalized adjunction formula, Math. Res. Letter 2 (1995), 453-471

[LL2] T. Li and A. Liu, Uniqueness of symplectic canonical class, surface cone and symplectic cone of 4-manifolds with $b^+ = 1$, Jour. Diff. Geom. 58 (2001), 331-370

[M] Moore, John D., Lectures on Seiberg-Witten Invariants, Springer-Verlag, 1996

[Mi] John Milnor, On Manifolds Homeomorphic to the 7-sphere, The Annals of Mathematics, 2nd Ser., Vol. 64, No. 2. (Sep., 1956), pp. 399-405

[Mu] James R. Munkres, Topology, 2nd Edition, Prentice Hall, Inc.

[MMS] J. Morgan, T Mrowka and Z. Szabó, Product formulas along T^3 for Seiberg-Witten invariants, Math. Research Letters 4 (1997), 915-929

[N] Liviu I. Nicolaescu, Notes on Seiberg-Witten Theory, GSM Volume 28, American Mathematical Society

[OV] C. Okonek and A. Van de Ven, Stable bundles and differentiable structures on certian algebra surfaces. Invent. Math. 86, 357-370 (1986)

[P1] Jongil Park, Simply Connected Symplectic 4-Manifolds with $b_2^+ = 1$ and $c_1^2 = 2$, 2003 arXiv:math.GT/0311395

[P2] Jongil Park, Exotic smooth structures on 4-manifolds. II, Topology Appl. 132 (2003) no.2, 195-202. MR 2004d:57033

[P3] Jongil Park, Exotic structures on $3\mathbb{CP}^2\#8\overline{\mathbb{CP}}^2$, arXiv:math.GT/0507085

[PA] B. Doug Park, Constructing infinitely many smooth structures on $3\mathbb{CP}^2\#n\overline{\mathbb{CP}}^2$, Math. Ann. 322 (2002), no.2, 267-278. MR 2003f:57061

[PSS] J. Park, A. Stipsicz and Z. Szabó, Exotic smooth structures on $\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$, Math. Res. Letters, arXiv:math.GT/0412216

[S] A. Scorpan, The Wild World of 4-Manifolds, American Mathematical Society, 2005

[Sa] Nikolai Saveliev, Lectures on the Topology of 3-Manifolds: An Introduction to the Casson Invariants, De Gruyter Textbook, 1999

[SS1] A. Stipsicz and Z. Szabó, An exotic smooth structure on $\mathbb{CP}^2\#6\overline{\mathbb{CP}}^2$, arXiv:math.GT/0411258

[SS2] A. Stipsicz and Z. Szabó, Small exotic 4-manifolds with $b_2^+ = 3$, arXiv:math.GT/0501273

[Sy1] M. Symington, Symplectic rational blowdowns, Jour. Diff. Geom. 50 (1998), 505-518

[Sy2] M. Symington, Generalized symplectic rational blowdowns, Alg. Gemo. Topo. 1 (2001), 503-518

[T1] C. Taubes, The Seiberg-Witten invariants and symplectic forms, Math. Res. Letters 1 (1994), 809-822

[T2] C. Taubes, More constraints on symplectic forms from Seiberg-Witten invariants, Math. Res. Letters 2 (1995), 9-13

[W] E. Witten, Monopoles and four-manifolds, Math. Research Letters 1 (1994), 769-796

[Y] Y. Matsumoto, An Introduction to Morse Theory, Translations of Mathematical Monographs Volume 208, American Mathematican Society